

COVERING  $\mathbb{R}$ -TREES

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ABSTRACT. We prove that every length space  $X$  is the orbit space (with the quotient metric) of an  $\mathbb{R}$ -tree  $\bar{X}$  via a free isometric action. In fact, for many well-known spaces, such as connected complete Riemannian manifolds of dimension at least two, the Menger sponge, and the Sierpin'ski gasket and carpet,  $\bar{X}$  is *the same* “universal”  $\mathbb{R}$ -tree  $A_c$ , which has valency  $c=2^{\aleph_0}$  at each point. The quotient mapping  $\bar{\phi} : \bar{X} \rightarrow X$  is a kind of generalized covering map called a URL-map, and  $\bar{X}$  is the unique (up to isometry)  $\mathbb{R}$ -tree that admits a URL-map onto  $X$ . The map  $\bar{\phi}$  is universal among URL-maps onto  $X$  and in fact is the “mother of all metric universal covers” in the following sense: All URL-maps, including the traditional universal cover of a semi-locally simply connected length space, may be naturally derived from it.

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## 1. INTRODUCTION AND MAIN RESULTS

In this paper we construct the *covering  $\mathbb{R}$ -tree*  $\bar{X}$  of a length space  $X$ , and the  $\mathbb{R}$ -tree universal covering map  $\bar{\phi} : \bar{X} \rightarrow X$ . The function  $\bar{\phi}$  is generally not a covering map in the traditional sense, but shares important properties with metric covering maps. Recall that if  $f : X \rightarrow Y$  is a traditional covering map and  $Y$  is a length space, the metric of  $Y$  may be “lifted” to  $X$  in a unique way that makes  $X$  a length space and  $f$  a local isometry. The function  $f$  has two additional basic properties: (I)  $f$  preserves the length of rectifiable paths in the sense that  $L(c) = L(f \circ c)$  for every path  $c$  in  $X$  with finite length  $L(c)$ . (II) If  $c$  is any rectifiable path in  $Y$  starting at a point  $p$  and  $f(q) = p$  then there is a unique path  $c_L$  starting at  $q$  such that  $f \circ c_L = c$ , and moreover  $c_L$  is rectifiable. A function  $f$  between length spaces will be called *unique rectifiable lifting (URL)* if  $f$  has these two properties. Note that a map between length spaces with condition (I) is known as an *arcwise isometry* ([16]). In fact any URL-map is a surjective open arcwise isometry. The class of URL-maps is closed under composition and, as we will see later, contains functions that are not local homeomorphisms—two essential features if one has as a goal finding generalizations of the traditional universal cover beyond spaces that are semi-locally simply connected. In the case of a regular (or normal) covering map  $f$ , the deck group  $G$  of the covering map acts freely via isometries on  $X$ , and  $Y$  is the metric quotient  $G \backslash X$ . In other words,  $Y$  is isometric to the orbit space  $G \backslash X$  with the Hausdorff metric on the orbits, and  $f$  is the corresponding quotient map. In particular, the traditional universal covering map  $\phi : \tilde{Y} \rightarrow Y$  (when it exists!) is regular with deck group  $\pi_1(Y)$ . Recall that  $\tilde{Y}$  is unique (up to isometry, with the lifted metric) and  $\phi$  has a universal property: if  $g : Z \rightarrow Y$  is a covering map then there is a unique (up to basepoint choice) covering map  $h : X \rightarrow Z$  such that  $\phi = g \circ h$ . For the following theorem we define  $\Lambda(X) := \lambda(X)/\eta(X)$ , where  $\lambda(X)$  is the group of rectifiable loops in the length space  $X$  starting at a given basepoint and  $\eta(X)$  is the normal subgroup of loops that are homotopic *in their image* to the trivial loop (see also Definition 15 and Proposition 20).

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**Theorem 1.** *For every length space  $X$  there exists a unique (up to isometry)  $\mathbb{R}$ -tree  $\overline{X}$ , called the covering  $\mathbb{R}$ -tree, with a URL-map  $\overline{\phi} : \overline{X} \rightarrow X$ . Moreover,*

- (1)  $\overline{X}$  is complete if and only if  $X$  is complete.
- (2) If  $Z$  is a length space and  $f : Z \rightarrow X$  is a URL-map then there is a unique (up to basepoint choice) URL-map  $\overline{f} : \overline{X} \rightarrow Z$  such that  $\overline{\phi} = f \circ \overline{f}$ .
- (3) The group  $\Lambda(X)$  acts freely via isometries on  $\overline{X}$  with metric quotient map  $\overline{\phi} : \overline{X} \rightarrow X = \Lambda(X) \backslash \overline{X}$ .

The term “ $\mathbb{R}$ -tree” was coined by Morgan and Shalen ([28]) in 1984 to describe a type of space that was first defined by Tits ([32]) in 1977. Originally  $\mathbb{R}$ -trees were defined as metric spaces with more than one point in which any two points are joined by a unique geodesic, i.e. an arclength parameterized curve with length equal to the distance between its endpoints. To avoid trivial special cases, for this paper define “length space” (resp. “geodesic space”) to be a metric space *with at least two points* such that any pair of points is joined by a path of length arbitrarily close to the distance between them (resp. joined by a geodesic). In the last three decades  $\mathbb{R}$ -trees have played a prominent role in topology, geometry, and geometric group theory (see, for example, [8], [28]). They are the most simple of geodesic spaces, and yet Theorem 1 shows that every length space, no matter how complex, is an orbit space of an  $\mathbb{R}$ -tree.

In 1928 Menger asked whether (in modern terminology) every Peano continuum (the continuous image of  $[0, 1]$ ) admits a compatible geodesic metric ([24]). More than 20 years later the problem was given a positive answer independently by Bing and Moise ([9], [26]). In fact, these two papers, together with earlier results of Menger, establish for compact, connected metric spaces the equivalence of (1) local connectedness, (2) local arcwise connectedness, and (3) the existence of a compatible geodesic metric. A few years later, R. D. Anderson announced ([2]) that every Peano continuum is the continuous image of the Menger sponge  $\mathbb{M}$  (the so-called “universal curve”) such that each point pre-image is also  $\mathbb{M}$ . A proof of Anderson’s theorem was eventually published by Wilson in 1972 ([37]), who at the same time proved a strengthened conjecture of Anderson ([3]) by showing that every Peano continuum is the image of a  $\mathbb{M}$  via a light open mapping (*light* means every point pre-image is totally disconnected). Anderson’s conjecture was part of a long-standing effort to construct dimension-raising open mappings, beginning with an example of Kolmogorov in 1937 ([20]) from a 1-dimensional Peano continuum to a 2-dimensional space (see also [19] for a dimension-raising light open mapping). Proposition 25 and the fact that a (non-trivial)  $\mathbb{R}$ -tree  $X$  is simply connected with small inductive dimension  $\text{ind}(X) = 1$  ([4]) give us:

**Corollary 2.** *Every non-trivial space admitting a compatible length metric is the image via a light open mapping of a simply connected space  $X$  with  $\text{ind}(X) = 1$ .*

Consider the following fractals: the Sierpin’ski carpet  $S_c$ , the Sierpin’ski gasket  $S_g$ , or  $\mathbb{M}$ . As is well-known, each such space  $X$  admits a geodesic metric  $d$  bi-Lipshitz equivalent to the metric induced by the metric  $\rho$  of the ambient Euclidean space;  $d(x, y)$  is defined as the infimum of the length of paths in  $X$  joining the points  $x$  and  $y$ , where the length is measured in the metric  $\rho$ . An  $\mathbb{R}$ -tree that appears very naturally in our work is the  $\mathfrak{c}$ -universal  $\mathbb{R}$ -tree  $A_{\mathfrak{c}}$  introduced in [23], which has valency  $\mathfrak{c} = 2^{\aleph_0}$  (cardinality of the continuum) at each point.  $A_{\mathfrak{c}}$  was shown in [23] to be metrically homogeneous, and “universal” in the sense that every  $\mathbb{R}$ -tree of valency at most  $\mathfrak{c}$  isometrically embeds in  $A_{\mathfrak{c}}$ . This is analogous to the original way in which  $\mathbb{M}$  was considered “universal”.

**Theorem 3.** *If  $X$  is a separable length space, then  $\overline{X}$  is a sub-tree of  $A_c$ . If in addition  $X$  is complete and contains a bi-Lipschitz copy of  $S_g$  or  $S_c$  at every point, e.g. if  $X$  is  $S_c$ ,  $S_g$ ,  $\mathbb{M}$ , or a complete Riemannian manifold of dimension at least two, then  $\overline{X}$  is isometric to  $A_c$ .*

Put another way, every separable length space may be obtained by starting with a subtree of  $A_c$  and taking a quotient of that subtree via a free isometric action. Another consequence of this theorem is an explicit construction of  $A_c$  starting with any of the above spaces (see the proof of Theorem 1). Our results, combined with the Anderson-Wilson Theorem, show that  $A_c$  is “universal” in another sense, which is similar to the second way in which  $\mathbb{M}$  may be regarded as “universal”:

**Corollary 4.** *Every Peano continuum is the image of  $A_c$  via a light open mapping.*

Finally, Theorem 1 provides yet a third (categorical) way in which  $A_c$  may be considered “universal”.

In addition to the properties described in the first paragraph, the traditional universal covering has two useful properties: the universal covering map is a fibration and the universal covering space is simply connected. These properties are used to classify traditional covering maps according to the subgroups of the fundamental group whose (representative) elements lift as loops. While  $\overline{X}$  is simply connected,  $\overline{\phi}$  is generally not a fibration. In fact, the only homotopies that may be lifted must already be “tree-like”, and it would be interesting to understand which homotopies do lift. For example, Piotr Hajlasz and Jeremy Tyson have constructed Lipschitz surjections from cubes onto compact, quasiconvex, doubling metric spaces ([22]), and it is easy to see from their construction that these mappings lift to the covering  $\mathbb{R}$ -tree. The mapping  $\overline{\phi}$  is not only *not* a fibration, it is in some sense as far as possible from being a fibration. This turns out to be an advantage. For any subgroup  $G$  of  $\Lambda(Y)$  we define a kind of “designer fibration” called a  $\text{URL}(G)$ -map  $f : X \rightarrow Y$ , which is a  $\text{URL}$ -map that lifts any (representative) loop in  $G$  as a loop. If the loops in  $G$  are the *only* loops that lift as loops then  $f$  is called “ $G$ -universal” (Definition 38). We prove:

**Theorem 5.** *Let  $X$  be a length space and  $G$  be a subgroup of  $\Lambda(X)$  considered as a group of isometries of  $\overline{X}$ . Suppose that the orbits of  $G$  are closed, the orbit space  $G \backslash \overline{X}$  is supplied with the quotient metric, and  $\psi_G : \overline{X} \rightarrow G \backslash \overline{X} := \overline{X}^G$  is the quotient mapping. Then there exists unique (continuous) map  $\overline{\phi}^G : \overline{X}^G \rightarrow X$  such that  $\overline{\phi}^G \circ \psi_G = \overline{\phi}$ . Moreover, if  $\overline{\phi}^G$  is a  $\text{URL}$ -map then*

- (1)  $\overline{X}^G$  is the unique (up to isomorphism)  $G$ -universal space with a  $\text{URL}(G)$ -map  $\overline{\phi}^G : \overline{X}^G \rightarrow X$  (also unique up to basepoint choice).
- (2) For any  $\text{URL}(G)$ -map  $f : Y \rightarrow X$  there is a unique (up to basepoint choice)  $\text{URL}$ -map  $f^G : \overline{X}^G \rightarrow Y$  such that  $\overline{\phi}^G = f \circ f^G$ .
- (3) If  $G$  is a normal subgroup of  $\Lambda(X)$  then the group  $\pi_1^G(X) := \Lambda(X)/G$  acts via isometries on  $\overline{X}^G$  and  $\overline{\phi}^G$  is the quotient mapping with respect to this action.
- (4)  $X$  is complete if and only if  $\overline{X}^G$  is complete.

When  $G$  is normal we call  $\pi_1^G(X)$  the  $G$ -fundamental group of  $X$ . The next corollary shows that  $\overline{\phi}^G$  is a true generalization of the traditional metric universal covering.

**Corollary 6.** *Let  $X$  be a semilocally simply connected length space. Then  $\overline{\phi}^{H_T}$  is the traditional metric universal covering of  $X$  and  $\pi_1^{H_T}(X)$  is naturally isomorphic to  $\pi_1(X)$ .*

In general, there is a natural homomorphism  $h_\Lambda : \Lambda(X) \rightarrow \pi_1(X)$ , the kernel of which is  $H_T$  and the image of which is the subgroup  $\mu_1(X)$  of  $\pi_1(X)$  consisting

of those homotopy classes having a rectifiable representative (see Proposition 40). Length spaces with bad (or unknown) local topology appear often in geometry and topology, from the classical examples such as the Menger sponge and Hawaiian earring to Gromov-Hausdorff limits of Riemannian manifolds. Very recently, Sormani and Wei have shown that such limits of manifolds with non-negative Ricci curvature have a universal covering in the categorical sense, but it is unknown whether this covering is simply connected ([33], [34]). Their work, in turn, was partly motivated by the 40-year old conjecture of Milnor that such non-positively curved manifolds have finitely generated fundamental groups ([25]). These papers and others ([38], [35], [36]) involve studying covering maps determined by geometrically significant groups of loops through a construction of Spanier ([31]). Our paper provides a much more general framework for such efforts. In fact, for any open cover  $\mathcal{U}$  of a connected, locally arcwise connected space, Spanier defined a covering space corresponding to a certain subgroup  $G_{\mathcal{U}}$  of the fundamental group. When  $X$  is a length space, Spanier's covering map is of the form  $\overline{\phi}^G$  where  $G := h_{\Lambda}^{-1}(G_{\mathcal{U}})$ .

In [6] we introduced the uniform universal covering (UU-covering)  $\phi : \tilde{X} \rightarrow X$ . The UU-covering is an analog of the universal covering for a large class of uniform spaces called *coverable* spaces, which includes all spaces admitting a length metric, and therefore all Peano continua. However, the UU-covering is not always satisfactory from a geometric standpoint. For example, the UU-coverings of the Hawaiian earring and the Menger sponge are connected but not arcwise connected, and the UU-covering of a non-compact semi-locally simply connected length space may not be its traditional universal covering (see [6]). In contrast, the covering  $\mathbb{R}$ -tree involves the more specialized (but extremely important) class of length spaces, but provides a way to construct a variety of generalized universal metric covering spaces that are always length spaces. Even so, the UU-covering proves useful in a couple of ways in the current paper. We use it to prove two essential theorems about paths (Theorems 10 and 11), and we use it to obtain additional examples of URL-maps that are not traditional covering maps. When  $X$  is a length space we show there exists a “metric core”  $\hat{X} \subset \tilde{X}$  that is a length space. Moreover, the restriction  $\hat{\phi} : \hat{X} \rightarrow X$  of the UU-covering map is a URL-map and a “metric fibration” in the sense that Lipschitz maps from geometrically reasonable simply connected domains may be lifted (Theorem 54). This may prove useful since in the case of the above-mentioned fractals and the compact separable infinite dimensional torus  $T^{\infty}$ , the metric core is a  $CAT(0)$  space even though the original space is not even locally simply connected. In fact the metric core of  $T^{\infty}$  is separable Hilbert space. In the case of a uniformly 1-dimensional length space,  $\overline{X}$  is naturally isometric to  $\hat{X}$  (Corollary 55). For length spaces (as opposed to uniform spaces in general), the construction (if not the proofs!) of the UU-covering is simpler to describe and we may use simplified notation. We give such a description and establish our notation in a short appendix.

## 2. THE COVERING $\mathbb{R}$ -TREE

The following are equivalent for a geodesic space  $X$  (see [27], [10], [4]): (1)  $X$  is an  $\mathbb{R}$ -tree. (2)  $X$  is 0-hyperbolic in Gromov's sense. (3)  $X$  contains more than one point and is  $CAT(K)$ -space for all  $K \leq 0$ . (4)  $X$  is simply connected and its small-inductive Urysohn-Menger dimension is 1. See, for example, [10] for the definitions of 0-hyperbolic and  $CAT(K)$ -space; we will not need the definitions in this paper. However, note that a corollary of Theorem 1 is that every length space is the metric quotient of a  $CAT(K)$  geodesic space for any  $K \leq 0$ .

A few words about dimension are in order. For a metric space  $X$  we denote the small (resp. large) inductive dimension by  $ind(X)$  (resp.  $Ind(X)$ ), and covering

dimension by  $\dim(X)$ . It is known that for an arbitrary metric space  $X$ , there are the Katetov equality  $Ind(X) = \dim(X)$  and the inequality  $ind(X) \leq Ind(X)$ , see [1]. If  $X$  is also separable (in particular compact) then  $ind(X) = Ind(X) = \dim(X)$ , [17]. Generally an  $\mathbb{R}$ -tree is not separable (for example,  $A_c$ ). For metric spaces (or more generally uniform spaces), one may also consider various definitions of “uniform dimension”. We use the same definition here as in [6], which is the same as the definition of covering dimension for a topological space except that the open covers involved are “uniform” in the following sense: An open cover of a metric space is uniform if it is refined by a cover of  $\varepsilon$ -balls for some  $\varepsilon > 0$ . This dimension is called “large dimension” in [18] and we will denote it by  $u\dim(X)$  in this paper. In general we do not know precisely the relationship between uniform dimension and the various topological notions of dimension.

**Notation 7.** *We will write “ $X$  has dimension  $\leq n$  in some sense (resp. in every sense)” if  $ind(X)$ ,  $Ind(X)$ ,  $\dim(X)$ , or (resp. and)  $u\dim(X)$  is at most  $n$ .*

For example, it is known that if  $X$  has dimension  $\leq n$  in some sense and  $C \subset X$  is compact then  $C$  has dimension  $\leq n$  in every sense. We will use this fact frequently below.

We next recall some basic background about paths, by which we mean continuous functions from compact intervals into metric spaces. While usage varies in the literature, we will say that two paths  $c_1, c_2 : I \rightarrow X$  are Fréchet equivalent if there exists an order-preserving homeomorphism  $c : I \rightarrow I$  such that  $c_1 = c_2 \circ c$ . If  $c$  is a path in a metric space we denote by  $L(c)$  its length, writing  $L(c) = \infty$  if  $c$  is not rectifiable. An arclength parameterized path is 1-Lipschitz, hence does not increase Hausdorff dimension (which is not smaller than covering dimension). It follows that the image of any nonconstant rectifiable path is a uniformly 1-dimensional Peano continuum. We will consider parameterizations only up to Fréchet equivalence, which simplifies many discussions. For example, given paths  $c_1$  and  $c_2$  defined on  $[0, L_1]$  and  $[0, L_2]$ , respectively, such that the starting point of  $c_2$  is the endpoint of  $c_1$ , we will simply refer to the concatenation  $c_1 * c_2$  on  $[0, L_1 + L_2]$  without mentioning the linear reparameterization of  $c_2$  to  $[L_1, L_1 + L_2]$  that is technically required to concatenate them. For any path  $c$  we will denote by  $c^{-1}$  the same path with orientation reversed (again using any convenient parameterization in the Fréchet equivalence class). When it comes to homotopies, we may say that two paths  $c_1$  and  $c_2$  are “endpoint-homotopic” even though, strictly speaking, there is only a homotopy between paths  $c'_1$  and  $c'_2$  that are Fréchet equivalent to  $c_1$  and  $c_2$ , respectively, that share a common domain.

Recall that in [12] a path  $c : [a, b] \rightarrow X$  in a metric space  $X$  is called *normal* if there is no nontrivial subsegment  $J = [u, v] \subset I$  such that  $c(u) = c(v)$ , and  $c|_J$  is homotopic to a constant relative to  $\{u, v\}$ .

**Definition 8.** *A path  $c : [a, b] \rightarrow X$  in a metric space  $X$  is called weakly normal if there is no nontrivial subsegment  $J = [u, v] \subset [a, b]$  such that  $c(u) = c(v)$  and  $c|_J$  is homotopic in  $c(J)$  to a constant relative to  $\{u, v\}$ . A weakly normal (rectifiable) arclength parameterized path (resp. loop)  $c : [a, a + L] \rightarrow X$  in a metric space  $X$  will be called a  $\rho$ -path (resp.  $\rho$ -loop).*

Put another way, a weakly normal path is one that is normal in its own image; clearly every normal path is weakly normal.

**Remark 9.** *In order to avoid special cases below, we permit the domain of a constant function to be an interval of the form  $[a, a] = \{a\}$ ; such a parameterization of a constant map is clearly a  $\rho$ -path.*

A version of the next theorem was proved for 1-dimensional separable metric spaces using very different methods (including Zorn's Lemma) in [12] (Lemma 3.1 and Theorem 3.1), also without considering lengths of paths. For our purposes it is very important to consider lengths and rectifiability. Moreover, as we have pointed out earlier, the  $\mathbb{R}$ -trees we will be considering may not be separable and therefore it is important to remove this assumption.

**Theorem 10.** *Each fixed-endpoint homotopy class of a path  $c : I \rightarrow X$  in metric space  $X$  that is 1-dimensional in some sense contains a unique (up to Fréchet equivalence) normal path  $c_n$ . In addition,  $L(c_n) \leq L(c)$ .*

*Proof.* First suppose that  $X$  is a Peano continuum. Then  $X$  has a UU-covering  $\tilde{X}$ . Choose as a basepoint  $*$  the starting point of  $c$  and suppose first that  $c$  is not a null-homotopic loop. By the lifting property of the UU-covering (see the Appendix), there is unique path  $\tilde{c} : I \rightarrow \tilde{X}$  starting at  $*$  such that  $\phi \circ \tilde{c} = c$ . Since  $X$  is compact,  $X$  is uniformly 1-dimensional and we may apply Proposition 60 to conclude that the function  $\lambda$  described in the appendix is injective. This means that  $\tilde{c}$  is a loop if and only if  $c$  is a null-homotopic loop. Therefore by our initial assumption  $\tilde{c}$  is not a loop. The image  $C = \tilde{c}(I)$  is a Peano continuum that contains no topological circle by Proposition 60. By the Hahn-Mazurkiewicz-Sierpin'ski Theorem,  $C$  is locally connected, hence a dendrite, see Section 51, VI of [21]. Then  $C$  is contractible by Corollary 7 in Section 54, VII of [21]. Hence there is a fixed-endpoint homotopy  $h : I \times I \rightarrow C$  such that  $h(\cdot, 0) = \tilde{c}$ , and  $h(\cdot, 1)$  is a topological embedding  $\tilde{c}_1$ , whose image is the unique arc  $a$  in  $C$  joining  $*$   $\in C$  and  $\tilde{c}(1) = \tilde{c}_1(1)$  (this arc exists by Corollary 7, Section 51, VI in [21]). By Proposition 60,  $a$  is also the unique arc in  $\tilde{X}$  joining  $\tilde{c}(1)$  and  $*$ . Then  $\phi \circ h$  is a fixed-endpoint homotopy in  $C$  from  $c$  to the unique (up to Fréchet equivalence) normal path  $c_n := \phi \circ \tilde{c}_1$ . To finish the proof in the case when  $c$  is not a null-homotopic loop, we need to prove that  $L(c_n) \leq L(c)$  if  $c$  is rectifiable. By the previous argument,  $a \subset C = \tilde{c}(I)$ , hence  $c_n(I) \subset c(I)$ . Let  $t_0 = 0 < t_1 < \dots < t_m = 1$  be a partition of  $I$  and  $x_0 = x = c_n(t_0), x_1 = c_n(t_1), \dots, x_m = c_n(t_m)$ . Then for any  $i = 1, \dots, m$  there is a maximal  $s_i \in I$  such that  $x_i = c(s_i)$ . It is clear that  $s_0 = 0 < s_1 < \dots < s_m = 1$ , so we get another partition of  $I$ . Now we have  $\sum_{i=0}^{m-1} d(c_n(t_i), c_n(t_{i+1})) = \sum_{i=0}^{m-1} d(c(s_i), c(s_{i+1})) \leq L(c)$ , and  $L(c_n) \leq L(c)$ .

If  $c$  is a non-constant null-homotopic loop then we may write it as the concatenation of two paths that are not null-homotopic loops and apply the case just considered; the path  $c_n$  in this case is constant. If  $c$  is constant, its homotopy class could not contain a non-constant normal path by the preceding sentence.

Returning to the general case for  $X$ , note that the image  $C$  of  $c$  is a Peano continuum that is 1-dimensional (with the subspace metric) in the same sense that  $X$  is, and hence we may apply the above special case to obtain the existence of the desired path  $c_n$  in the fixed-endpoint homotopy class of  $c$  in  $C$ . Now suppose that  $c'$  is any normal path in the fixed-endpoint homotopy class of  $c$  in  $X$ . Let  $h$  be a homotopy from  $c_n$  to  $c'$ . We may apply a similar argument using the compact image  $K$  of  $h$  to conclude from the uniqueness in the case of a Peano continuum that  $c_n$  and  $c'$  are Fréchet equivalent.  $\square$

**Theorem 11.** *A path  $c : I \rightarrow X$  in a metric space that is 1-dimensional in some sense is normal if and only if  $c$  is weakly normal.*

*Proof.* As in the proof of Theorem 10, we may assume that  $X$  is a Peano continuum. We have already observed that every normal path is weakly normal. Let us suppose that the path  $c$  is not normal. Then there is a nontrivial subsegment  $J = [u, v] \subset I$  such that  $c(u) = c(v)$ , and  $c|J$  is homotopic to a constant ( $c(u) = c(v)$ ) relative to

$\{u, v\}$ . Let  $h : J \times I \rightarrow X$  be the corresponding homotopy. By the lifting property, there is a unique homotopy  $\tilde{h} : J \times I \rightarrow \tilde{X}$  such that  $\tilde{h}(u, 0) = \tilde{c}(u)$  and  $h = \phi \circ \tilde{h}$ . Then  $\phi \circ \tilde{h}(u, s) = \phi \circ \tilde{h}(v, s) = c(u) = c(v)$  for all  $s \in I$ , and by Remark 59,  $\tilde{h}(u, s) = \tilde{c}(u)$  and  $\tilde{h}(v, s) = \tilde{c}(v)$  for all  $s \in I$ . Evidently,  $\tilde{h}(t, 1)$  is constant for all  $t \in J$ , so  $\tilde{c}(u) = \tilde{c}(v)$ . Then the restriction  $\tilde{c}|J$  is a loop in  $\tilde{X}$  and its image  $C = \tilde{c}(J)$  is a dendrite as in the proof of Theorem 10. There is a fixed-endpoint homotopy  $\tilde{h}_1 : J \times I \rightarrow C$  such that  $\tilde{h}_1(\cdot, 0) = \tilde{c}|J$  and  $\tilde{h}_1(t, 1) \equiv \tilde{c}(u) = \tilde{c}(v)$  for all  $t \in J$ . Then  $h_1 = \phi \circ \tilde{h}_1$  gives a fixed-endpoint homotopy in  $c(J)$  to the constant path  $c(u) = c(v)$ . This means that the path  $c$  is not weakly normal.  $\square$

From our previous observation that the image of a rectifiable path is a 1-dimensional Peano continuum we may conclude:

**Corollary 12.** *Let  $X$  be a metric space and  $c$  be path in  $X$ , with image  $C$  that is 1-dimensional in some sense (resp. rectifiable). There is a unique weakly normal (resp.  $\rho$ -path)  $\gamma : [0, L(\gamma)] \rightarrow C$  that is fixed-endpoint homotopic to  $c$  in  $C$ . In addition,  $L(c) \geq L(\gamma)$ .*

**Corollary 13.** *In any length space, for every pair of points  $p, q$ ,  $d(p, q)$  is the infimum of the lengths of injective  $\rho$ -paths joining  $p$  and  $q$ .*

Note that geodesics are injective and hence normal.

**Corollary 14.** *If  $X$  is a simply connected length space that is 1-dimensional in some sense (in particular if  $X$  is an  $\mathbb{R}$ -tree) then every pair of points is joined by a unique  $\rho$ -path, which is also a geodesic (hence the unique geodesic) joining them. Moreover, this  $\rho$ -path is contained in the image of every path joining the two points.*

*Proof.* There is such a path, since any two points  $p, q$  are connected by a  $\rho$ -path. Moreover, since  $X$  is simply connected, any two  $\rho$ -paths joining them must be fixed-endpoint homotopic, hence by Theorems 10 and 11, there can only be one  $\rho$ -path joining them. Finally,  $d(p, q)$  is equal to the infimum of the lengths of  $\rho$ -paths joining them—but up to Fréchet equivalence there is only one such path, so the infimum must be a minimum.  $\square$

Note that the concatenation of a  $\rho$ -path  $c$  followed by a  $\rho$ -path  $d$  may not be a  $\rho$ -path. To resolve this problem we define the “cancelled concatenation”  $c \star d$  to be the unique  $\rho$ -path in the fixed-endpoint homotopy class of the concatenation  $c * d$ , in the image of  $c * d$  (Corollary 12). By uniqueness, we see more concretely that  $c \star d$  is obtained from  $c * d$  by removing the maximal final segment of  $c$  that coincides with an initial segment of  $d$  with reversed orientation, and removing that initial segment of  $d$  as well. Also by uniqueness, the associative law  $(a \star b) \star c = a \star (b \star c)$  is satisfied. Moreover, cancelled concatenation on  $\rho$ -loops at a fixed basepoint is a group operation, where the constant loop is the identity and the inverse of  $c$  is the  $\rho$ -loop  $c^{-1}$  already discussed above.

**Definition 15.** *Let  $X$  be a length space with basepoint  $*$  and define  $(\overline{X}, *)$  (or simply  $\overline{X}$  when no confusion with regard to basepoint will result) to be the set of all  $\rho$ -paths  $c : [0, L(c)] \rightarrow X$  starting at  $*$  (i.e.  $c(0) = *$ ), and let  $\overline{\phi} : \overline{X} \rightarrow X$  be the endpoint mapping. We place the following metric on  $\overline{X}$ : Let  $c_1 \wedge c_2 : [0, b] \rightarrow X$  be the restriction of  $c_1$  (and  $c_2$ ) to the largest interval  $[0, b]$  on which  $c_1$  and  $c_2$  coincide. For  $c_1, c_2 \in \overline{X}$ , define*

$$(1) \quad d_{\overline{X}}(c_1, c_2) := L(c_1) + L(c_2) - 2L(c_1 \wedge c_2) = L(c_1^{-1} \star c_2)$$

*We let  $\Lambda(X, *) \subset (\overline{X}, *)$  (or simply  $\Lambda(X)$ ) denote the group of all  $\rho$ -loops starting at  $*$ , with the subspace metric.*

It is easy to check that  $d_{\overline{X}}$  is a metric.

**Proposition 16.** *For a length space  $X$  with basepoints  $*$  and  $*'$ , any  $\rho$ -path  $k$  from  $*'$  to  $*$  induces an isometry from  $(\overline{X}, *)$  to  $(\overline{X}, *')$  defined by  $c \mapsto k \star c$ , and composition of this isometry with the endpoint mapping coincides with  $\overline{\phi}$ .*

*Proof.* In fact, for  $\rho$ -curves  $c_1, c_2$  starting at  $*$ , the previously discussed associative law implies

$$d_{\overline{X}}(k \star c_1, k \star c_2) = L((k \star c_1)^{-1} \star (k \star c_2)) = L(c_1^{-1} \star c_2) = d_{\overline{X}}(c_1, c_2)$$

□

That is, up to isometry,  $\overline{\phi} : \overline{X} \rightarrow X$  is independent of choice of basepoint in  $X$ . For this reason we will often avoid discussion of basepoints and simply assume that all functions are basepoint preserving, in particular choosing the constant path  $*$  as the basepoint in  $\overline{X}$ .

**Corollary 17.** *The group  $\Lambda(X)$  acts freely as isometries on  $(\overline{X}, *)$  (hence on  $\Lambda(X)$ ) by left multiplication.*

**Lemma 18.** *For any length space  $X$ , the metric on  $\Lambda(X)$  is complete, hence  $\Lambda(X)$  is a complete topological group.*

*Proof.* First note that it is an immediate consequence of the definition of the metric that for any  $c, d \in \overline{X}$ ,

$$|L(c) - L(d)| \leq d(c, d)$$

Let  $(c_i)$  be a Cauchy sequence in  $\Lambda(X)$  with  $c_i$  parameterized on  $[0, L_i]$ . By the above inequality,  $(L_i)$  is a real sequence converging to a non-negative real number  $L$ . Now for any  $L' < L$  the loops  $c_i$  are defined on  $[0, L']$  and equal, for all sufficiently large  $i$ . We may define  $c : [0, L] \rightarrow X$  by  $c(t) = c_i(t)$  for all large  $i$ , and  $c(L) = *$ . It is easy to check that this defines an element of  $\Lambda(X)$ , and certainly  $c_i \rightarrow c$  in the metric of  $\Lambda(X)$ . It is a standard result that a group with a complete left-invariant metric is complete as a topological group. □

**Remark 19.** *It should be observed that the isometry from Proposition 16 does not take loops to loops and so does not induce an isometry from  $\Lambda(X, *)$  to  $\Lambda(X, *')$ . In fact, these groups are in general isomorphic as groups, but not isometric. An isomorphism from  $\Lambda(X, *)$  to  $\Lambda(X, *')$  can be obtained by choosing any  $\rho$ -path  $c$  from  $*'$  to  $*$  and taking  $d \in \Lambda(X, *)$  to  $c \star d \star c^{-1} \in \Lambda(X, *')$ . To see that these groups need not be isometric, let  $X$  consist of a circle of circumference 1 having a segment  $[a, b]$  of length 1 glued to it at  $a$ . The shortest distance from a non-trivial element of  $\Lambda(X, a)$  to the trivial loop at  $a$  is 1, but the shortest distance from a non-trivial element of  $\Lambda(X, b)$  to the trivial loop at  $b$  is 2.*

One can easily prove the following:

**Proposition 20.** *Let  $\lambda(X)$  be the collection of all rectifiable loops at the basepoint in  $X$  with concatenation as the group operation. Define  $h : \lambda(X) \rightarrow \Lambda(X)$  by letting  $h(c)$  be the unique  $\rho$ -path in the fixed-endpoint homotopy class of  $c$  in the image of  $c$ . Then  $h$  is a surjective homomorphism with kernel equal to the (normal) subgroup  $\eta(X)$  of all null-homotopic in itself rectifiable loops. That is,  $\Lambda(X) = \lambda(X)/\eta(X)$ .*

For the next theorem we need to recall some definitions (see the survey article [29] for more details). Submetries were introduced by the first author as a generalization of the notion of Riemannian submersion (see [5] and references there). Recall that if  $X, Y$  are metric spaces,  $f : X \rightarrow Y$  is a submetry (resp. weak submetry) if for every closed ball  $B(p, r)$  (resp. open ball  $U(p, r)$ ) in  $X$ ,  $f(B(p, r)) = B(f(p), r)$



(resp.  $f(U(p, r)) = U(f(p), r)$ ). There are two trivially equivalent conditions that we will use without further comment: (1) The function  $f$  is distance non-increasing (i.e. 1-Lipschitz) and for every  $x, y \in Y$  and  $z \in f^{-1}(x)$  (resp. and any  $\varepsilon > 0$ ) there exists  $w \in f^{-1}(y)$  such that  $d(z, w) = d(x, y)$  (resp.  $d(z, w) < d(x, y) + \varepsilon$ ). (2) For every  $x, y \in X$  and  $z \in f^{-1}(x)$ ,  $d(x, y) = \min\{d(z, w) : w \in f^{-1}(y)\}$  (resp.  $d(x, y) = \inf\{d(z, w) : z \in f^{-1}(x) \text{ and } w \in f^{-1}(y)\}$ ). An easy consequence of the definition is that  $f$  is open. Obviously weak submetries do not increase the lengths of paths. It follows that if  $f : X \rightarrow Y$  is a weak submetry and  $X$  is a length space then so is  $Y$ , although  $Y$  need not be a geodesic space even if  $X$  is. Every isometry is obviously a submetry; on the other hand any injective weak submetry is an isometry. If a group  $G$  acts on a metric space  $M$  by isometries then the orbit space  $G \backslash M$  may be given the quotient pseudometric, namely the Hausdorff distance between the orbits (which may be 0 for different orbits if the two orbits are not closed). In fact,  $G \backslash M$  is a metric space if and only if the orbit  $Gx = \{g(x) : g \in G\}$  of every  $x \in M$  is closed. Since  $G$  acts by isometries and is transitive on each orbit, we also have  $d(Gx, Gy) = \inf\{d(g(x), y) : g \in G\}$ . It follows from the second characterization above that with respect to this metric the *metric quotient mapping*  $\phi : M \rightarrow G \backslash M$  is a weak submetry.

**Theorem 21.** *For any length space  $X$ ,  $\overline{X}$  is an  $\mathbb{R}$ -tree and  $\overline{\phi}$  is the metric quotient mapping (hence a weak submetry) with respect to the isometric action of  $\Lambda(X)$ .*

*Proof.* We will show that  $\overline{X}$  is an  $\mathbb{R}$ -tree using the characterization (2) from the beginning of this section. We will also denote by  $*$  the element of  $\overline{X}$  that is simply the constant path at  $*$   $\in X$ . Let  $c_1, c_2 \in \overline{X}$ , defined on  $[0, L_1]$ ,  $[0, L_2]$ , respectively. Let

$$s_0 := \max\{s : c_1(t) = c_2(t) \text{ for all } t \in [0, s]\}$$

and define  $C(s)$  for  $s \in [0, L_1 + L_2 - 2s_0]$  as follows. For  $s \in [0, L_1 - s_0]$  let  $C(s)$  be the restriction of  $c_1$  to  $[0, L_1 - s]$ . For  $s \in [L_1 - s_0, L_1 + L_2 - 2s_0]$  let  $C(s)$  be the restriction of  $c_2$  to  $[0, s - L_1 + 2s_0]$ . Certainly  $C(s)$  is a geodesic in  $\overline{X}$ , joining  $c_1$  and  $c_2$ . This implies that  $\overline{X}$  is a geodesic space.

We see from Formula (1) that the so-called Gromov product

$$(c_1, c_2)_* := \frac{1}{2}[d_{\overline{X}}(*, c_1) + d_{\overline{X}}(*, c_2) - d_{\overline{X}}(c_1, c_2)]$$

with respect to the point  $*$  (see, for example, [10], page 410) is equal to  $L(c_1 \wedge c_2)$ . Also we see immediately that  $c_1 \wedge c_2$  contains as a subpath  $(c_1 \wedge c_3) \wedge (c_2 \wedge c_3)$  for any  $c_3$ . Then it follows from these two statements that

$$(2) \quad (c_1, c_2)_* \geq \min\{(c_1, c_3)_*, (c_3, c_2)_*\}$$

for any  $c_1, c_2, c_3 \in \overline{X}$ . This means that  $\overline{X}$  is 0-hyperbolic “with respect to the point  $*$ ”, whereas 0-hyperbolicity itself means that the equation (2) must be always satisfied if we replace  $*$  by any path-point  $c$ . By Remark 1.21 (page 410 of [10]),  $\overline{X}$  is actually 0-hyperbolic, so it is an  $\mathbb{R}$ -tree.

To show that  $\overline{\phi}$  is the quotient mapping we need to check: (1) For any  $x \in X$ ,  $\overline{\phi}^{-1}(x)$  is precisely the orbit  $\Lambda(X)y$  of any point  $y \in \overline{\phi}^{-1}(x)$  and (2) for any  $x, w \in X$ ,  $d(x, w)$  is equal to the Hausdorff distance  $d_H(\overline{\phi}^{-1}(x), \overline{\phi}^{-1}(w))$ . Note that if  $\overline{\phi}(c) = \overline{\phi}(d)$  then by definition  $c$  and  $d$  have the same endpoint. So  $c \star d^{-1} \in \Lambda(X)$  and  $c = (c \star d^{-1}) \star d$ , which means that  $c$  and  $d$  lie in the same orbit. Conversely, if  $c$  and  $d$  lie in the same orbit,  $c = k \star d$  for some  $k \in \Lambda(X)$ , so both curves have the same endpoint. This proves condition (1). Now by definition, for any  $c, f \in \overline{X}$ ,  $d(c, f)$  is at least the distance between their endpoints  $p, q$ , respectively, since  $d(c, f)$  is the length of a curve joining  $p$  and  $q$ . This implies that  $d_H(\overline{\phi}^{-1}(p), \overline{\phi}^{-1}(q)) \geq d(p, q)$ . On the

other hand, given any  $p, q \in X$  and  $\varepsilon > 0$  there is a  $\rho$ -path  $c$  from  $*$  to  $p$  and a  $\rho$ -path  $k$  from  $p$  to  $q$  such that  $L(k) \leq d(p, q) + \varepsilon$ . Let  $d := c \star k$ . Then  $d(c, f) = L(f^{-1} \star c) \leq L(k) \leq d(p, q) + \varepsilon$ . Letting  $\varepsilon \rightarrow 0$  we see  $d_H(\bar{\phi}^{-1}(p), \bar{\phi}^{-1}(q)) \leq d(p, q)$ .  $\square$

**Proposition 22.** *If  $f : X \rightarrow Y$  is a URL-map then  $X$  is complete if and only if  $Y$  is complete.*

*Proof.* We shall prove necessity and sufficiency simultaneously. Suppose that  $X$  (resp.  $Y$ ) is complete. Let  $\{y_i\}$  be a Cauchy sequence in  $Y$  (resp.  $X$ ). Choose a subsequence  $\{y_{i_j}\}$  so that  $d(y_{i_j}, y_{i_{j+1}}) < 2^{-j}$ . Let  $\gamma_j$  be an arclength parameterized rectifiable path in  $Y$  (resp.  $X$ ) from  $y_{i_j}$  to  $y_{i_{j+1}}$  such that  $L(\gamma_j) \leq 2^{-j+1}$  and  $\Gamma_j := \gamma_1 \star \dots \star \gamma_{j-1}$ . Take any  $x_1 \in f^{-1}(y_{i_1})$  (resp.  $x_1 = f(y_{i_1})$ ) and for  $j > 1$  define  $x_j$  to be the endpoint of the lift  $(\Gamma_j)_L$  of  $\Gamma_j$  starting at  $x_1$  (resp. of the path  $f \circ \Gamma_j$ ). Since these lifts preserve length (resp.  $f$  is an arcwise isometry), it follows by the triangle inequality that  $d(x_j, x_{j+1}) \leq 2^{-j+1}$  and hence  $\{x_j\}$  is Cauchy, hence convergent to a point  $x \in X$  (resp.  $x \in Y$ ). It is clear that there exists a unique arclength parameterized path  $\Gamma$  joining  $x_1$  to  $x$  such that for sufficiently large  $j$ ,  $\Gamma$  coincides with the paths  $(\Gamma_j)_L$  (resp.  $f \circ \Gamma_j$ ) on any closed subsegment of the domain not containing the right endpoint. Then the subsequence  $\{y_{i_j}\}$ , hence the sequence  $y_i$ , converges to the endpoint of the path  $f(\Gamma)$  (resp. the lift of  $\Gamma$  starting at  $y_{i_1}$ ).  $\square$

**Lemma 23.** *A map  $f : X \rightarrow Y$  between length spaces is a URL-map if and only if  $f$  is 1-Lipshitz and for some choice of basepoints,  $f$  is basepoint preserving, each arclength parameterized rectifiable path  $p$  starting at the basepoint has unique lift  $p_L$  starting at the basepoint, and  $L(p) = L(p_L)$ .*

*Proof.* The necessity of these conditions easily follows from the definition of URL-map. Let us prove sufficiency. Assume first that  $c$  is an arclength parameterized rectifiable curve starting at  $y \in Y$  and  $x \in X$  satisfies  $f(x) = y$ . Let  $k$  be a  $\rho$ -curve from the basepoint in  $X$  to  $x$ . Then  $d := f \circ k$  is rectifiable and since each of its initial segments has, by assumption, a unique lift of the same length,  $d$  is also arclength parameterized and has the same length as  $k$ . Then  $d \star c$  is rectifiable and arclength parameterized, so has a unique lift  $(d \star c)_L$  to the basepoint in  $X$ . We may write  $(d \star c)_L = k \star k'$  for some curve  $k'$ . Then  $k'$  is the desired lift of  $c$ ;  $k'$  must be unique since if it were not then  $d \star c$  would not have a unique lift. Now suppose that  $c : [0, a] \rightarrow Y$  is a rectifiable path starting at  $y$ , and  $f(x) = y$ . Let  $\mathcal{C}$  be the collection of maximal (closed) intervals on which  $c$  is constant. As is well-known, there are a non-decreasing continuous function  $h : [0, a] \rightarrow [0, L(c)]$  and an arclength parameterized rectifiable path  $c_1 : [0, L(c)] \rightarrow Y$  such that  $c = c_1 \circ h$  and  $h$  is strictly increasing everywhere except on the intervals in  $\mathcal{C}$ . Let  $d_1$  be the unique lift of  $c_1$  at  $x$ . Define  $c_L : [0, a] \rightarrow X$  by  $c_L(t) = d_1 \circ h(t)$ . Then  $c_L$  has the same length as  $d_1$ , hence as  $c_1$  and  $c$ .

Since  $f$  is 1-Lipshitz, it now follows from the lifting property proved above that if  $c$  is rectifiable in  $X$  then  $f \circ c$  has the same length as  $c$ . If  $c$  is not rectifiable then  $f \circ c$  cannot be rectifiable either, for if it were,  $f \circ c$  would have a rectifiable lift and a non-rectifiable lift.  $\square$

**Lemma 24.** *Every URL-map  $f : X \rightarrow Y$  is a weak submetry. If  $Y$  is a geodesic space then  $f$  is a submetry.*

*Proof.* Since  $Y$  is a length space, for any  $\varepsilon > 0$  and  $x, y \in Y$  we may join  $x, y$  by a rectifiable path  $c$  with the length less than  $d(x, y) + \varepsilon$ . By definition,  $c$  has a lift  $c_L$  of the same length with endpoints  $w, z$  such that  $f(w) = x$  and  $f(z) = y$ . Since  $X$  is a length space,  $d(w, z) \leq L(c_L) = L(c) \leq d(x, y) + \varepsilon$  and the proof of the first part is complete. Let suppose that  $Y$  is a geodesic space,  $x$  is a fixed, and  $y$  is arbitrary points

in  $Y$ . Then the points  $x, y$  can be joined by a (shortest) geodesic  $\gamma$ . For any point  $z \in f^{-1}(x)$ , there exists a unique lift  $\bar{\gamma}$  of  $\gamma$ , starting at  $z$ , having the same length as  $\gamma$  and some endpoint  $w$ . Then  $f(w) = y$ ,  $d(z, w) \leq L(\bar{\gamma}) = L(\gamma) = d(x, y)$ . On the other hand, it follows from the first part that  $f$  is 1-Lipschitz, so  $d(x, y) \leq d(z, w)$  and  $d(x, y) = d(z, w)$ . This implies that  $f$  is a submetry.  $\square$

*Proof of Theorem 1.* The existence of  $\bar{X}$  and the fact that  $\bar{\phi}$  is a weak submetry, hence 1-Lipschitz, follow from Theorem 21. To show that  $\bar{\phi}$  is URL, suppose first that  $c : [0, L] \rightarrow X$  is a  $\rho$ -curve at the basepoint. Let  $\gamma_c$  denote the unique geodesic in  $\bar{X}$  from the basepoint  $*$  to  $c$ . The following formula is easy to check:

$$(3) \quad c = \bar{\phi} \circ \gamma_c$$

In particular,  $\gamma_c$  is a lift of  $c$  to  $\bar{X}$  starting at the basepoint, and, being a geodesic, is a  $\rho$ -path. Since both curves are arclength parameterized,  $L(c) = L(\bar{\phi} \circ \gamma_c)$ . Because  $c$  is weakly normal, any lift  $c_L$  of  $c$  also has to be weakly normal and hence, by Corollary 14, the unique geodesic  $\gamma_d$  joining  $*$  to some  $d \in \bar{X}$ . But Formula (3) implies that  $c = d$  and hence  $c_L = \gamma_c$ .

Now suppose that  $d$  is a  $\rho$ -curve starting at  $z \in X$  and  $k \in \bar{X}$  satisfies  $\bar{\phi}(k) = z$ ; that is,  $k = \bar{\phi} \circ \gamma_k$  is a  $\rho$ -curve from the basepoint to  $z$ . By what we just proved,  $c := k \star d$  has a unique lift  $c_L$  to  $\bar{X}$  starting at the basepoint. Now it is possible that there is some maximal final segment  $b$  of  $k$  that, with reversed orientation, coincides with some initial segment of  $d$ ; let  $c''$  be the remaining segment of  $d$ . Finally, let  $s$  be the final segment of  $\gamma_k$  such that  $\bar{\phi} \circ s = b$  and  $h$  be the final segment of  $c_L$  that maps onto  $c''$ . Then it is easy to check that  $d_L := s^{-1} \star h$  is the unique lift of  $d$  starting at  $k$ , and  $L(d_L) = L(d)$ .

By Lemma 23, to finish the proof that  $\bar{\phi}$  is URL we need only consider an arbitrary arclength parameterized rectifiable curve  $c : [0, L] \rightarrow X$  starting at the basepoint. By Corollary 12 we have, for every  $0 \leq s < t \leq a$ , a unique  $\rho$ -path  $\rho_{s,t} : [s, t] \rightarrow C := c([0, L])$  such that  $\rho_{s,t}$  is fixed-endpoint homotopic to  $c_{s,t} := c|_{[s,t]}$  and  $L(\rho_{s,t}) \leq L(c_{s,t}) = t - s$ . Define  $c_L(t)$  to be the endpoint of the unique lift  $\rho_t^L$  of  $\rho_{0,t}$  to  $\bar{X}$  starting at the basepoint, which exists by the special case proved above. Obviously  $c_L$  is a lift of  $c$ . By uniqueness for any  $s < t$  we must have  $\rho_{0,t} = \rho_{0,s} \star \rho_{s,t}$ . Since we have shown above that  $\rho_{s,t}$  has a unique lift to  $c_L(s)$  of the same length as  $\rho_{s,t}$ , which by uniqueness must end at  $c_L(t)$ , we have

$$(4) \quad d(c_L(s), c_L(t)) \leq L(\rho_{s,t}) \leq t - s.$$

In other words,  $c_L : [0, L] \rightarrow \bar{X}$  is 1-Lipschitz and  $L(c_L) \leq L$ . Since we already know that  $\bar{\phi}$  is distance non-increasing, it follows that  $L(c_L) \geq L(c) = L$ . Therefore  $c_L$  is a length-preserving lift of  $c$ . Now suppose that  $d$  is any lift of  $c$  starting at the basepoint. For any  $t$  let  $\gamma_t$  be the geodesic from the basepoint to  $d(t)$ , which according to Corollary 14 is contained in the image of  $d$  and is fixed-endpoint homotopic to  $d$ . Now  $\bar{\phi} \circ \gamma_t$  is the  $\rho$ -curve  $d(t) \in \bar{X}$ , which lies in the image of  $\bar{\phi} \circ d = c$  and is fixed-endpoint homotopic to  $c$ . By uniqueness,  $\bar{\phi} \circ \gamma_t = \rho_{0,t}$ . Therefore  $c_L(t) = d(t)$  for all  $t$  and the proof that  $\bar{\phi}$  is URL is finished.

Now Part (1) of the theorem follows from Proposition 22.

The uniqueness of  $\bar{X}$  will follow from Lemma 24 once we have proved the second part of the theorem. Given a URL-map  $f : Z \rightarrow X$ , with some choice of basepoints, define  $\bar{f}(c)$  to be the endpoint of the unique lift of  $c$  starting at the basepoint in  $Z$ . Obviously  $f \circ \bar{f} = \bar{\phi}$ . For  $c, k \in \bar{X}$ , the lift of  $c \star k^{-1}$  is a path joining  $f(c)$  and  $f(k)$  having the same length as  $c \star k^{-1} = d(c, k)$ , and therefore  $\bar{f}$  is 1-Lipschitz. Now let  $\gamma$  be a rectifiable path starting at the basepoint in  $Z$ . Then  $f \circ \gamma$  is rectifiable in  $X$ , so has a lift  $(f \circ \gamma)_L$  at the basepoint in  $\bar{X}$ . Now  $\bar{f} \circ (f \circ \gamma)_L$  is a lift of  $f \circ \gamma$  starting at

the basepoint in  $Z$  and so must be equal to  $\gamma$ . That is,  $(f \circ \gamma)_L$  is a lift of  $\gamma$  starting at the basepoint in  $\overline{X}$  having the same length as  $\gamma$ . Suppose  $k$  is any lift of  $\gamma$  starting at the basepoint in  $\overline{X}$ . Then  $\gamma$  is also a lift of  $f \circ \gamma$  to  $\overline{X}$  and therefore  $k = (f \circ \gamma)_L$ . We have checked the conditions of Lemma 23 to show that  $\overline{f}$  is a URL-map. Finally, suppose we have a URL-map  $h$  that preserves the basepoints with  $f \circ h = \overline{\phi}$ . For any  $c \in \overline{X}$ ,  $h \circ \gamma_c$  is a rectifiable path from the basepoint to  $h(c)$ , which is also a lift of  $\overline{\phi} \circ \gamma_c = c$  starting at the basepoint in  $Z$ . Since this lift is unique,  $h(c) = \overline{f}(c)$ .  $\square$

### 3. CONTINUA, FRACTALS, AND MANIFOLDS

**Proposition 25.** *For any length space,  $\overline{\phi}$  is light.*

*Proof.* As is well-known, any connected subset of an  $\mathbb{R}$ -tree is arcwise connected. Now if  $\overline{\phi}^{-1}(p)$  contained a connected subset with more than one point then that subset would contain a geodesic with constant image, a contradiction to uniqueness of lifting of rectifiable paths.  $\square$

In [23] an  $\mathbb{R}$ -tree  $T$  was defined to be  $\mu$ -universal for a cardinal number  $\mu$  if every  $\mathbb{R}$ -tree of valency at most  $\mu$  isometrically embeds in  $T$ . Recall that for a point  $t \in T$  the valency at  $T$  is the cardinality of the set of connected components of  $T \setminus \{t\}$ , and  $T$  is said to have valency at most  $\mu$  if the valency of every point in  $T$  is at most  $\mu$ . The existence, uniqueness (up to isometry), and homogeneity of universal  $\mathbb{R}$ -trees for a given cardinal number  $\mu$  was proved in [23]. In fact, the  $\mu$ -universal  $\mathbb{R}$ -tree is uniquely the complete  $\mathbb{R}$ -tree with valency  $\mu$  at each point. In [15] a more explicit construction was given for the universal  $\mu$ -universal  $\mathbb{R}$ -tree  $A_\mu$  for arbitrary cardinal number  $\mu \geq 2$ .

**Definition 26.** *Define  $\rho$ -paths  $\alpha : [0, a] \rightarrow X$  and  $\beta : [0, b] \rightarrow X$  starting at  $p$  to be equivalent if  $\alpha, \beta$  coincide on  $[0, \varepsilon)$  for some  $\varepsilon > 0$ . We denote the cardinality of the set of the resulting equivalence classes by  $\kappa_p$ .*

**Proposition 27.** *Let  $X$  be a length space, and  $p \in X$ . The valency of  $\overline{X}$  at any point  $\overline{p} \in \overline{\phi}^{-1}(p)$  is equal to  $\kappa_p$ . If  $X$  is separable then  $\kappa_p \leq \mathfrak{c} = 2^{\aleph_0}$ .*

*Proof.* The construction of  $\overline{X}$  immediately implies the first statement. If  $X$  is separable, then  $X$  itself has cardinality  $\mathfrak{c}$  (unless it is a point). Since every path is determined by its value at rational numbers in its domain, then the cardinality of  $\kappa_p$  is at most  $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$ .  $\square$

**Example 28.** *If we consider the space  $B$  consisting of countably many circles of radius 1 attached at a point  $p$  then there is a natural geodesic metric that is uniformly one-dimensional, even though the space is not compact. The valency of points in  $\overline{B}$  is either  $\aleph_0$  or 2 depending on whether they are in  $\overline{\phi}^{-1}(p)$  or not.*

The algebraic topology of one-dimensional continua, which are known to be  $K(\pi, 1)$  spaces (cf. [13]), has been studied since the 1950's. See, for example, [11] for a good bibliography, recent work and open questions concerning the fundamental groups of such spaces. A basic example to consider is the Hawaiian earring  $H$ , metrized so that it consists of countably many circles  $\{C_i\}_{i \in \mathbb{N}}$  each of length  $2^{1-i}$ , all attached at a common basepoint  $*$ , and given the induced geodesic metric.

**Proposition 29.** *In the Hawaiian earring  $H$  we have  $\kappa_p = \mathfrak{c}$  for the point  $p = *$  and  $\kappa_p = 2$  for any point  $p \neq *$ .*

*Proof.* The last statement is evident. It is easy to find  $\mathfrak{c}$  unit weakly normal loops starting (and ending) at  $p = *$  such that no two coincide on any interval  $[0, \varepsilon)$ . In fact, one can define a path that wraps one of two ways around  $C_1$ , then one of two

ways around  $C_2$ , and so on. Then reverse the parametrization, so that  $C_1$  is wrapped around last. It is clear that any such path is encoded as a sequence  $\{x_n\}_{n \in \mathbb{N}}$  with values in the set  $\{-1, 1\}$ . An arclength parameterized path will be equivalent to one of these if and only if it wraps the same way around  $C_i$  for all sufficiently large  $i$ , or in other words, defines an equivalent sequence  $\{y_n\}$ —we mean here that  $x$  is equivalent to  $y$  if there is a number  $m \in \mathbb{N}$  such that  $x_n = y_n$  for all  $n \geq m$ . The following lemma finishes the proof.  $\square$

**Lemma 30.** *In the Hawaiian earring  $H$ , we have  $\mathfrak{c}$  different equivalence classes of the above type of sequence.*

*Proof.* Let us take first the sequence  $x_n \equiv 1$ . Consider now an arbitrary sequence  $z = \{z_n\}$  of natural numbers and define subsequently two sequences  $s(z)_n := \sum_{i=1}^n z_i$  and  $\sigma(z)_n := \sum_{i=1}^n 2^{s(z)_i}$ . It is clear that if for another such sequence  $w = \{w_n\}$ ,  $w_m \neq z_m$ , then  $\sigma(w)_n \neq \sigma(z)_n$  for all  $n \geq m$ . Then for every sequence  $z$  above, define another sequence  $x(z)_n$  with values in  $\{-1, 1\}$  by the equations  $x(z)_m = -1$  if  $m = \sigma(z)_k$  for some  $k \in \mathbb{N}$ , and  $x(z)_m = 1$  for all other  $m \in \mathbb{N}$ . Then it follows from the statement above that  $x(z)_n$  is not equivalent to  $x$  and is not equivalent to  $x(w)$  if  $w \neq z$ . So we get  $\aleph_0^{\aleph_0} = \mathfrak{c}$  pairwise non-equivalent sequences of above sort.  $\square$

*Proof of Theorem 3.* The first part of the theorem is an immediate consequence of Proposition 27 and the theorem from [23] that every  $\mathbb{R}$ -tree of valency at most  $\mathfrak{c}$  isometrically embeds in  $A_{\mathfrak{c}}$ . Next, for any point  $p \in S_{\mathfrak{c}}$  there is clearly a bi-Lipschitz embedding  $h : H \rightarrow S_{\mathfrak{c}}$  such that  $h(*) = p$ . Therefore  $\kappa_p \geq \mathfrak{c}$ . The case of  $S_g$  is more tricky. There are countably many rectifiable loops  $C_i$  in  $S_g$  starting at any fixed point  $p$  such that every  $C_i$  is a topologically embedded circle and  $L(C_{i+1}) < \frac{1}{3}L(C_i)$  for all natural numbers  $i$ . Then one can prove with a little more details than in Proposition 29, that  $\kappa_p \geq \mathfrak{c}$ . A similar argument now finishes the proof of the theorem.  $\square$

#### 4. URL( $G$ )-MAPS

Throughout this section, let  $X$  and  $Y$  be length spaces. The following statement easily follows from definitions.

**Proposition 31.** *The collection of pointed length spaces with URLs as morphisms is a category, which we will refer to as the URL category.*

**Proposition 32.** *Let  $f : (X, *) \rightarrow (Y, *)$  be a basepoint preserving URL-map of length spaces. Then there is a commutative diagram*

$$\begin{array}{ccc} (\overline{X}, *) & \xrightarrow{\overline{f}} & (\overline{Y}, *) \\ \downarrow \overline{\phi_X} & & \downarrow \overline{\phi_Y} \\ (X, *) & \xrightarrow{f} & (Y, *) \end{array}$$

*of URL-maps preserving basepoints, with unique  $\overline{f}$ , where  $\overline{\phi_X}$  and  $\overline{\phi_Y}$  are  $\mathbb{R}$ -tree covering maps for  $X$  and  $Y$  respectively, and  $\overline{f}$  is isometry.*

*Proof.* By Theorem 1, there are unique basepoint preserving URL-maps  $\psi : (\overline{Y}, *) \rightarrow (X, *)$  such that  $f \circ \psi = \overline{\phi_Y}$ , and  $\overline{f} : (\overline{X}, *) \rightarrow (\overline{Y}, *)$  such that  $\psi \circ \overline{f} = \overline{\phi_X}$ . Then  $\overline{\phi_Y} \circ \overline{f} = f \circ \psi \circ \overline{f} = f \circ \overline{\phi_X}$ . By Proposition 31, the composition  $f \circ \overline{\phi_X}$  is a URL-map. It follows from Theorem 1 that there exists a unique URL-map  $\overline{g} : (\overline{Y}, *) \rightarrow (\overline{X}, *)$  such that  $f \circ \overline{\phi_X} \circ \overline{g} = \overline{\phi_Y}$ . By the previous argument,  $\overline{\phi_X} \circ \overline{g} = \psi$ , and  $\overline{\phi_X} \circ \overline{g} \circ \overline{f} = \psi \circ \overline{f} = \overline{\phi_X}$ . Now Theorem 1 and the previous equations imply that  $\overline{g} \circ \overline{f} = 1_{\overline{Y}}$ . By a similar argument,  $\overline{\phi_Y} \circ (\overline{f} \circ \overline{g}) = f \circ \overline{\phi_X} \circ \overline{g} = f \circ \psi = \overline{\phi_Y}$ , and  $\overline{f} \circ \overline{g} = 1_{\overline{X}}$ . Thus  $\overline{f}$  and  $\overline{g}$

are inverses of one another, and both weak submetries (hence, 1-Lipschitz maps) by Proposition 24. Then both  $\overline{f}$  and  $\overline{g}$  are (surjective) isometries.  $\square$

So in the situation of Proposition 32,  $(\overline{X}, *)$  can be identified with  $(\overline{Y}, *)$ .

**Lemma 33.** *If  $f : X \rightarrow Y$  is URL then a path  $c$  in  $X$  is a  $\rho$ -path if and only if  $f \circ c$  is a  $\rho$ -path.*

*Proof.* Identifying  $(\overline{Y}, *)$  with  $(\overline{X}, *)$  via the isometry  $\overline{f}^{-1}$ , we get that  $f \circ \overline{\phi_X} = \overline{\phi_Y}$  (see Proposition 32). This implies that  $c$  and  $f \circ c$  have the same lift  $\gamma_c$  to  $\overline{X}$  with respect to the URL-maps  $\overline{\phi_X}$  and  $\overline{\phi_Y}$ . As in the proof of Theorem 1 either of the curves  $c$  or  $f \circ c$  is  $\rho$ -curve if and only if  $\gamma_c$  is a geodesic in  $\overline{X}$ . This finishes the proof.  $\square$

**Proposition 34.** *If  $f : X \rightarrow Y$  is URL then letting  $f_*(c) := f \circ c$  defines an injective homomorphism  $f_* : \Lambda(X) \rightarrow \Lambda(Y)$ . In addition, the orbits of the subgroup  $f_*(\Lambda(X))$  in  $\overline{Y}$  are closed, and  $\phi_X$  and  $f$  are naturally identified, respectively, with the metric quotient map  $\psi : \overline{Y} \rightarrow f_*(\Lambda(X)) \backslash \overline{Y}$  and the map  $\phi : f_*(\Lambda(X)) \backslash \overline{Y} \rightarrow \Lambda(Y) \backslash \overline{Y}$  induced by the inclusion map  $f_*(\Lambda(X)) \rightarrow \Lambda(Y)$ .*

*Proof.* If  $c$  is the cancelled concatenation of  $c_1$  and  $c_2$  in  $\Lambda(X)$ , then by uniqueness  $f \circ c$  is the cancelled concatenation of  $f \circ c_1$  and  $f \circ c_2$ , which is precisely what it means for  $f_* : \Lambda(X) \rightarrow \Lambda(Y)$  to be a homomorphism. The injectivity of  $f_*$  immediately follows from definitions and Lemma 33. Now if we identify  $\overline{X}$  with  $\overline{Y}$  via the isometry  $\overline{f}$ , then  $\Lambda(X)$  is naturally identified with  $f_*(\Lambda(X))$  while the lift of any element  $c \in \Lambda(X)$  with respect to  $\overline{\phi_X} : \overline{X} \rightarrow X$  is identified with the lift of  $f \circ c$  with respect to  $\overline{\phi_Y}$ , because  $f \circ \overline{\phi_X} = \overline{\phi_Y}$ . This, together with Lemma 24, implies the other statements of the proposition.  $\square$

**Definition 35.** *A URL-map  $f : (X, *) \rightarrow (Y, *)$  is called URL( $G$ ) for some subgroup  $G$  of  $\Lambda(Y)$  if for every  $c \in G$ , its lift  $c_L$  to  $X$  starting at  $*$  is a loop.*

If  $H$  is a subgroup of  $G$  then obviously every URL( $G$ )-map is a URL( $H$ )-map. By Lemma 33, the lift  $c_L$  in the above definition is in fact a  $\rho$ -loop. Proposition 34 immediately implies:

**Corollary 36.** *If  $f : X \rightarrow Y$  is URL then:*

- (1) *If  $G$  is a subgroup of  $\Lambda(Y)$  and  $f$  is URL( $G$ ) then  $f^*(G) := \{c_L : c \in G\}$  is a subgroup of  $\Lambda(X)$ , isomorphic to  $G$ .*
- (2) *If  $H$  is a subgroup of  $\Lambda(X)$  then  $f_*(H) := \{f \circ c \in \Lambda(Y) : c \in H\}$  is a subgroup of  $\Lambda(Y)$ , isomorphic to  $H$ .*

**Example 37.** *Suppose  $f : (X, *) \rightarrow (Y, *)$  is a traditional covering map,  $X$  is connected,  $Y$  is a length space, and  $X$  has the lifted metric. Since traditional metric covering maps are URL-maps and fibrations,  $\pi$  is URL( $H_T$ ), where  $H_T$  is the group of null-homotopic  $\rho$ -loops in  $Y$  at  $*$ .*

**Definition 38.** *If  $G$  is a subgroup of  $\Lambda(Y)$  then  $X$  is called  $G$ -universal if there is a URL( $G$ ) function  $f : X \rightarrow Y$  such that if  $c$  is a  $\rho$ -loop at  $*$  in  $X$  then  $f \circ c \in G$ . In this case the map  $f$  will also be called  $G$ -universal.*

From Proposition 34 we have:

**Proposition 39.** *A URL-map  $f : X \rightarrow Y$  is URL( $G$ ) if and only if  $G \subset f_*(\Lambda(X))$ , and  $G$ -universal if and only if  $f_*(\Lambda(X)) = G$ .*

**Proposition 40.** *The map  $h_\Lambda : \Lambda(X) \rightarrow \pi_1(X)$  that takes each  $\rho$ -loop to its homotopy equivalence class, is a homomorphism with the kernel  $H_T$  and image  $\mu_1(X)$ , the subgroup of  $\pi_1(X)$  consisting of all equivalence classes having at least one rectifiable representative. Therefore, if each element of the fundamental group of  $X$  contains a rectifiable representative, then  $\pi_1(X)$  is naturally isomorphic to  $\Lambda(X)/H_T := \pi_1^{H_T}(X)$ .*

*Proof.* The map  $h_\Lambda$  is homomorphism, since the canceled concatenation of two  $\rho$ -curves is fixed-endpoint homotopic to their concatenation. Evidently, its kernel is  $H_T$ ; the statement about image follows from Corollary 12.  $\square$

**Remark 41.** *It is not true in general, even for compact geodesic spaces  $X$ , that each element of the fundamental group of  $X$  contains a rectifiable representative (cf. [7]), contrary to Remark 1.13 (b) on p. 10 in [16].*

*Proof of Corollary 6.* Let  $\pi : Y \rightarrow X$  be the traditional universal covering map. Moreover, since  $Y$  is simply connected, the image of every loop in  $Y$  is an element of  $H_T$ , and this means  $\pi$  is  $H_T$ -universal. From Proposition 34 it follows that  $Y$  is naturally isometric to  $H_T \backslash \overline{X}$ . It is well-known (and easy to check) that since  $X$  is semi-locally simply connected, every path contains a piecewise geodesic in its fixed-endpoint homotopy class, so the second part of the corollary follows from Proposition 40.  $\square$

We will need the following lemma, which is a kind of metric Second Isomorphism Theorem.

**Lemma 42.** *Let  $G$  act by isometries on  $X$  and  $f : X \rightarrow Y = G \backslash X$  be the quotient map. Suppose  $H$  is a subgroup of  $G$  and  $\psi : X \rightarrow Z = H \backslash X$  is the quotient map.*

- (1) *There is a unique function  $\phi : Z \rightarrow Y$  such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Z \\ \searrow f & & \swarrow \phi \\ & Y & \end{array}$$

- (2) *If  $H$  has closed orbits and  $Z$  is given the quotient metric then  $\phi$  is a weak submetry.*  
 (3) *If  $H$  is a normal subgroup of  $G$  with closed orbits then  $G/H$  acts naturally as isometries on  $Z$  and  $\phi$  is the quotient mapping onto  $Y = (G/H) \backslash Z$ .*

*Proof.* Define  $\phi(\psi(x)) = f(x)$ ; this is the only possible way to define  $\phi$  to make the diagram commute, so we need only check that it is well-defined. But if  $\psi(x) = \psi(y)$  then by definition both  $x$  and  $y$  lie in the same orbit of  $H$ , hence the same orbit of  $G$ , so  $f(x) = f(y)$ . For the second part note that  $\phi$  is distance non-increasing because the orbits of  $H$  are contained in the orbits of  $G$ . Let  $x, y \in Y$  and  $\varepsilon > 0$ . Since  $f$  is a weak submetry there are  $a \in f^{-1}(x), b \in f^{-1}(y)$  such that  $d(a, b) < d(x, y) + \varepsilon$ . Then by definition  $d(Ha, Hb) \leq d(a, b) < d(x, y) + \varepsilon$ ,  $\phi(Ha) = x$  and  $\phi(Hb) = y$ , which proves that  $\phi$  is a weak submetry.

For the third part define  $gH(Hx) = Hg(x)$  for  $x \in X$  and  $g \in G$ . Since  $H$  is normal it is easy to check that we have a well-defined function corresponding to the coset  $gH \in G/H$ . Given  $g_1, g_2 \in G$ ,

$$\begin{aligned} (g_1 g_2) H(Hx) &= Hg_1(g_2(x)) = g_1 H(Hg_2(x)) \\ &= g_1 H(g_2 H(Hx)) = (g_1 H \circ g_2 H)(Hx). \end{aligned}$$

That is, we have a properly defined action. Note that this also implies that each function  $gH$  has an inverse and hence is injective.

Moreover, since  $d(Hx, Hy) = \inf\{d(k(x), y) : k \in H\}$  and  $g$  is an isometry,

$$d(Hg(x), Hg(y)) = \inf\{d(kg(x), g(y)) : k \in H\}$$

$$= \inf\{d(g^{-1}kg(x), y) : k \in H\} = d(Hx, Hy)$$

Now  $Hx, Hy$  lie in the same orbit of this action if and only if  $Hg(x) = Hy$  for some  $g \in G$ ; equivalently  $x$  and  $y$  lie in the same orbit of  $G$ , which is equivalent to  $f(x) = f(y)$ . By definition this is equivalent to  $\phi(Hx) = \phi(Hy)$ . This shows that the orbits of the action of  $G/H$  are the same as the inverse images of  $\phi$ . Finally we need to check that  $Y$  has the quotient metric with respect to this action. But the distance between the orbits of  $Hx$  and  $Hy$  is the infimum of the distances between  $Hx$  and  $Hg(y)$ , where  $g \in G$ . This in turn is the infimum of distances between  $x$  and  $kg(y)$ , where  $k \in H$ , which is the distance between the orbits of  $x$  and  $y$  with respect to  $G$ . But since  $Y$  has the quotient metric, this is precisely  $d(f(x), f(y)) = d(\phi(Hx), \phi(Hy))$ .  $\square$

**Notation 43.** For any subgroup  $G$  of  $\Lambda(Y)$  that has closed orbits in  $\overline{Y}$  we will denote  $G \backslash \overline{Y}$  with the quotient metric by  $\overline{Y}^G$ . We will use the notation  $\psi_G : \overline{Y} \rightarrow G \backslash \overline{Y}$  for the quotient mapping and  $\overline{\phi}^G : \overline{Y}^G \rightarrow Y = \Lambda(Y) \backslash \overline{Y}$  for the mapping analogous to  $\phi$  from Lemma 42.

**Proposition 44.** Let  $G$  be a subgroup of  $\Lambda(Y)$  and suppose  $f : X \rightarrow Y$  is a  $URL(G)$ -map. Then

- (1) There is a unique (up to basepoint choice) function  $g : \overline{Y}^G \rightarrow X$  such that  $\overline{\phi}^G = f \circ g$ , and moreover  $g$  is a weak submetry if  $G$  has closed orbits in  $\overline{Y}$ .
- (2) If  $f$  is  $G$ -universal then  $G$  has closed orbits in  $\overline{Y}$ ,  $g$  is an isometry, and  $f$  can be identified with  $\phi_G$ .

*Proof.* Using Proposition 32 and Proposition 34 we identify  $\Lambda(X)$  with  $f_*(\Lambda(X)) \subset \Lambda(Y)$  and  $\phi_X$  with  $\psi : \overline{Y} \rightarrow f_*(\Lambda(X)) \backslash \overline{Y}$ . Proposition 39 implies that  $G \subset f_*(\Lambda(X))$ , and from Lemma 42 we have a unique (up to basepoint choice) mapping  $g : \overline{Y}^G \rightarrow f_*(\Lambda(X)) \backslash \overline{Y} = X$ , which is a weak submetry when the orbits of  $G$  are closed. With these identifications,  $f \circ g = \phi_G$ , and uniqueness follows from the fact that these identifications are all uniquely determined by the basepoint choice.

The second part follows from Proposition 39, since in this case  $G = f_*(\Lambda(X))$ ,  $g : \overline{Y}^G \rightarrow f_*(\Lambda(X)) \backslash \overline{Y}$  is the identity map, and under our identifications  $f$  is identified with  $\phi_G$ .  $\square$

$URL(G)$ -maps have the following useful property:

**Proposition 45.** Let  $X, Y, Z$  be length spaces and assume that the following diagram of continuous base-preserving maps is commutative:

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ & \searrow f & \swarrow g \\ & Y & \end{array}$$

If  $g$  is a  $URL$ -map, then  $f$  is a  $URL$ -map if and only if  $h$  is a  $URL$ -map, and each of the maps  $f$  or  $h$  is uniquely defined by the another one and  $g$ .

Suppose further that  $G$  is a subgroup of  $\Lambda(Y, *)$ , and  $g$  is  $URL(G)$ .

- (1) If  $h$  is  $URL(g^*(G))$  then  $f$  is  $URL(G)$ .
- (2) If  $f$  is  $URL(G)$  then  $h$  is a uniquely determined (up to basepoint choice)  $URL(g^*(G))$ -map.

*Proof.* In checking that maps are  $URL$  we will use Lemma 23 without further reference, and all paths will be assumed to start at the basepoint. Suppose first that  $h$  is  $URL$ . Then  $f = g \circ h$  is  $URL$  by Proposition 31. We must have  $f = g \circ h$ , so  $f$  is uniquely determined by  $h$  and  $g$ .



Now suppose that  $f$  is URL and  $c$  is a path in  $Z$  of the length  $L < \infty$ . Then  $c_Y := g \circ c$  has length  $L$ , and thus has unique lift  $c_X$  to  $X$  of length  $L$ . Then  $g \circ h \circ c_X = f \circ c_X = c_Y$  and  $g \circ c = c_Y$ , so  $c = h \circ c_X$  because  $g$  is URL. Thus  $c_X$  is a lift of  $c$ . If there is another such lift  $c'_X$ , then  $f \circ c'_X = g \circ h \circ c'_X = g \circ c = c_Y$  and  $c'_X = c_X$ , because  $c_Y$  is a rectifiable path and  $f$  is URL. Moreover,  $L(h \circ c_X) = L(g \circ (h \circ c_X)) = L(f \circ c_X) = L(c)$  because  $g$  and  $f$  are URL. Thus  $h$  preserves the length of paths, and  $h$  is URL. Let  $x \in X$  and  $\gamma$  be any  $\rho$ -curve from  $*$  to  $x$  in  $X$ . Then  $h \circ \gamma$  must be the unique lift of  $f \circ \gamma$  to  $Z$  and therefore  $h(x)$  is uniquely determined by  $f$  and  $g$  (and the basepoints).

Suppose now that  $G$  is a subgroup of  $\Lambda(Y, *)$ , and  $g$  is URL( $G$ ).

(1) If  $h$  is URL( $g^*(G)$ ) and  $c \in G$  then since  $g$  is URL( $G$ ),  $c_L \in g^*(G)$  ( $c_L$  is the lift of  $c$  to  $Z$ ), and hence lifts as a loop  $d$  in  $X$ . However, by a similar argument to the above,  $d$  is the unique lift of  $c$  to  $X$  (we have already shown  $f$  is URL!) and this shows that  $f$  is URL( $G$ ).

(2) If  $f$  is URL( $G$ ) and  $c \in g^*(G)$  then the lift  $d$  of  $g \circ c \in G$  to  $X$  is a loop. Now  $g \circ h \circ d = f \circ d = g \circ c$  and hence  $h \circ d$  is the unique lift of  $g \circ c$  to  $Z$ . But then  $c = h \circ d$ . That is, the loop  $d$  is the unique lift of  $c$  to  $X$ , which proves that  $h$  is URL( $g^*(G)$ ).  $\square$

**Remark 46.** *Similar arguments show that in the setting of Proposition 45,  $g$  is URL provided both  $f$  and  $h$  are URL and  $g$  has the property that for any non-rectifiable path  $c$  in  $Z$ ,  $g \circ c$  is non-rectifiable.*

*Proof of Theorem 5.* The first statement follows from Lemma 42 and Theorem 21. Since  $\overline{\phi}^G$  is URL by assumption, similar to the proof of Proposition 44, there exists unique mapping  $\psi : \overline{X} \rightarrow \overline{X}^G$  such that  $\overline{\phi}^G \circ \psi = \overline{\phi}$ . Then  $\psi = \psi_G$  and it follows from Proposition 32 that  $\overline{X}$  and  $\psi_G$  are naturally identified respectively with  $\overline{X}^G$  and the  $\mathbb{R}$ -tree universal covering map for  $\overline{X}^G$ . It is clear now by Proposition 34 that  $\overline{\phi}_*^G(\Lambda(\overline{X}^G)) = G$ , and  $\overline{\phi}^G$  is  $G$ -universal by Proposition 39.

The second part of the theorem follows by taking  $f^G := g$  from Proposition 44 (with  $X$  and  $Y$  interchanged) and observing that  $f^G$  is uniquely determined and URL by Proposition 45. If  $f$  is  $G$ -universal, then by Proposition 44 (2),  $f^G$  is an isometry. This implies the uniqueness of  $\overline{\phi}^G$  as a  $G$ -universal map onto  $X$ , and together with the above considerations, the first part of the theorem.

The third part is a consequence of Lemma 42. The fourth part follows from Proposition 22 because  $\overline{\phi}^G$  is URL.  $\square$

**Remark 47.** *Following the classical usage, it now makes sense to say that a URL-map  $f : X \rightarrow Y$  is called normal (or regular) if  $f_*(\Lambda(X))$  is a normal subgroup of  $\Lambda(Y)$ . We may then refer to  $\Lambda(Y)/f_*(\Lambda(X))$  as the group of deck transformations of  $f$ .*

**Example 48.** *Let  $X$  be the Euclidean plane, and let  $c$  and  $d$  be different semicircles on the same circle of circumference 2 parameterized as  $\rho$ -paths with the same starting point  $*$  and endpoint  $x$ . Take sequences  $c(t_i) \rightarrow x$  and  $d(t_i) \rightarrow x$  with  $t_i$  strictly increasing. Define  $c_i := (d|_{[0, t_i]}) \star f \star (c|_{[0, t_i]})^{-1}$  where  $f$  is an arclength parameterized straight line from  $d(t_i)$  to  $c(t_i)$ . Let  $G$  be the subgroup of  $\Lambda(X)$  generated by the loops  $c_i$ . Note that no loop in  $G$  passes through the point  $x$ , and therefore for any  $k \in G$ ,  $d(k, d \star c^{-1}) > 1$ . In other words,  $d \star c^{-1}$  is not in the closure of  $G$ . This shows that simply being a closed subgroup of  $\Lambda(X)$  is not sufficient to have closed orbits.*

**Lemma 49.** *If  $G$  is a subgroup of  $\Lambda(X)$  then the orbits of  $G$  are closed in  $\overline{X}$  if and only if  $G$  is strongly closed the following sense: Given  $c_i \in G$  and  $c \in \overline{X}$  such that  $c_i \star c \rightarrow d$  then  $d \star c^{-1} \in G$ .*

*Proof.* That the orbits of  $G$  are closed means precisely that if  $c \in \overline{X}$  and  $c_i \in G$  such that  $c_i \star c \rightarrow d$ , then there is some  $a \in G$  such that  $d = a \star c$ . If  $d \star c^{-1} \in G$  then certainly it plays the role of  $a$ . On the other hand, if such an  $a$  exists then since  $a$  is a loop,  $c$  and  $d$  have the same endpoint. Then we may concatenate on each side to obtain  $d \star c^{-1} = a$ .  $\square$

**Remark 50.** *If  $G$  is strongly closed then given a convergent sequence  $(c_i)$  in  $G$ , we may apply the strongly closed criterion with  $c$  the trivial loop to see that the limit of  $(c_i)$  is actually in  $G$ , implying that  $G$  is closed.*

The next Proposition shows that for any strongly closed group  $G$ , the mapping  $\overline{\phi}^G$  is tantalizingly close to being URL.

**Proposition 51.** *If  $G$  is any strongly closed subgroup of  $\Lambda(X)$  then  $\overline{\phi}^G : \overline{X}^G \rightarrow X$  is a weak submetry such that*

- (1) *For every  $x, y \in \overline{X}^G$ ,  $d(x, y)$  is the infimum of the lengths of rectifiable paths  $c$  such that  $\overline{\phi}^G \circ c$  is a  $\rho$ -path in  $X$  of the same length as  $c$ .*
- (2) *For every rectifiable path  $k$  starting at  $p \in X$  and  $q \in (\overline{\phi}^G)^{-1}(p)$  there is some lift  $d$  of  $k$  to  $\overline{X}^G$  of the same length as  $k$  and starting at  $q$ .*

*Proof.* That  $\overline{\phi}^G$  is a weak submetry follows from Lemma 42. Since  $\psi_G$  is a quotient map, for  $x, y \in \overline{X}^G$  and  $\varepsilon > 0$  we may find a geodesic  $\gamma$  joining points  $w, z$  with  $\psi_G(w) = x$ ,  $\psi_G(z) = y$ , and  $d(x, y) \leq L(\gamma) < d(x, y) + \varepsilon$ . Let  $c := \psi_G \circ \gamma$ . Since  $\psi_G$  and  $\overline{\phi}^G$  are both distance (hence length) nonincreasing,

$$L(c) \geq L(\overline{\phi}^G \circ c) = L(\overline{\phi} \circ \gamma) = L(\gamma) \geq L(c)$$

Letting  $\varepsilon \rightarrow 0$  finishes the proof of the first part.

For the second part, let  $q' \in \overline{X}$  be such that  $\psi_G(q') = q$ , let  $\gamma_k$  be the lift of  $k$  to  $q'$  (which is a geodesic) and let  $d := \psi_G \circ \gamma_k$ . Since  $\overline{\phi}^G \circ \psi_G = \overline{\phi}$ ,  $d$  is a lift of  $k$ , and by the same argument as in the first part,  $d$  has the same length as  $k$ .  $\square$

A basic question remains: whether  $\overline{\phi}^G : \overline{X}^G \rightarrow X$  from Proposition 51 is a URL-map for every strongly closed subgroup of  $\Lambda(X)$ ?

## 5. THE METRIC CORE

In this section we construct a large family of URL-maps for spaces that may not be semi-locally simply connected. In many cases the maps involve domains that are simply connected, and in some cases even  $CAT(0)$ . For this section the reader is referred to the appendix for notation and background. Let  $H_L$  denote the group of  $\rho$ -paths that are null-homotopic via a 1-Lipschitz homotopy. In this section let  $X$  and  $Y$  be length spaces.

**Definition 52.** *For  $x, y \in \tilde{X}$ , define*

$$d(x, y) := \inf\{L(\phi \circ \gamma)\}$$

*where the infimum is taken over all paths  $\gamma$  joining  $x$  and  $y$ , and we take the convention that the infimum of the empty set is  $\infty$ . Define*

$$\hat{X} := \left\{x \in \tilde{X} : d(x, *) < \infty\right\}.$$

The set  $\widehat{X}$  will be called the metric core of  $\widetilde{X}$  and we will denote the restriction of  $\phi$  to  $\widehat{X}$  by  $\widehat{\phi}$ , and the restrictions of the projections  $\phi_i$  will be denoted by  $\widehat{\phi}_i$ ; we will also denote  $\widehat{\phi}$  by  $\widehat{\phi}_0$ .

**Definition 53.** Let  $\mu(X) := \lambda(\mu_1) \subset \delta(X)$ . We will call  $\mu(X)$  the uniform metric deck group of  $X$ .

**Theorem 54.** The function  $d$  defined above defines a (finite) length space metric on  $\widehat{X}$  such that

- (1) The inclusion of  $\widehat{X}$  with this metric into  $\widetilde{X}$  is uniformly continuous.
- (2) If  $Y$  is a length space and  $g : Y \rightarrow \widetilde{X}$  is a function such that  $g(Y) \cap \widehat{X} \neq \emptyset$  and  $\phi \circ g$  is 1-Lipschitz then  $g(Y) \subset \widehat{X}$  and  $g$  is 1-Lipschitz (hence continuous) in the metric of  $\widehat{X}$ .
- (3) Every 1-Lipschitz homotopy between rectifiable paths in  $X$  lifts to  $\widehat{X}$  and in particular  $\widehat{\phi}$  is an  $H_L$ -URL-map.
- (4) If  $X$  is complete then  $\widehat{X}$  is complete.
- (5) The group  $\mu(X)$  is the stabilizer of  $\widehat{X}$  in the uniform fundamental group  $\delta(X)$  and acts isometrically and freely on  $\widehat{X}$  with metric quotient  $\mu(X) \backslash \widehat{X} = X$ .

*Proof.* We will use Notation 58 from the appendix. As mentioned in the appendix, for any  $i$ ,  $\phi_i : \widetilde{X} \rightarrow X_i$  is the UU-covering of  $X_i$  and therefore this mapping has the same lifting properties as  $\phi$ . Symmetry, positive definiteness, and the triangle inequality (with possibly infinite values and the usual conventions for adding extended real numbers) of  $d$  are clear from the definition. From this it follows that if  $x, y \in \widehat{X}$  then  $d(x, y) < \infty$ , hence  $d$  is a (finite) metric on  $\widehat{X}$ . The fact that  $\widehat{X}$  is a length space will follow from the definition of the metric if we show that  $\widehat{\phi}$  is length-preserving. Since the distance between any two points in  $X$  is the infimum of lengths of paths joining them, including paths that may not be projections of paths in  $\widetilde{X}$ ,  $\widehat{\phi}$  is distance non-increasing and hence length non-increasing. If  $\widehat{\phi} \circ \gamma : [0, 1] \rightarrow X$  is rectifiable then for any partition  $t_0, \dots, t_k$  of  $[0, 1]$  we have by definition of the distance in  $\widehat{X}$ ,

$$\sum_{i=1}^k d(\gamma(t_i), \gamma(t_{i-1})) \leq \sum_{i=1}^k L(\widehat{\phi} \circ \gamma|_{[t_{i-1}, t_i]}) = L(\widehat{\phi} \circ \gamma) < \infty$$

and therefore  $\gamma$  is rectifiable and  $L(\gamma) \leq L(\widehat{\phi} \circ \gamma)$ . Therefore  $\widehat{\phi}$  is length-preserving. A similar argument shows that every  $\widehat{\phi}_i$  is similarly length-preserving, a fact that we will need below.

To show that the inclusion is uniformly continuous, due to the compatibility properties of  $\widetilde{d}$  discussed in the appendix (the uniform structure is the inverse limit structure), we need only show the following: For any  $\varepsilon > 0$  there exists some  $\delta > 0$  such that if  $x, y \in \widehat{X}$  satisfy  $d(x, y) < \delta$  then  $d(\phi_i(x), \phi_i(y)) < \varepsilon$  for any  $i$ . But in fact each  $\phi_i$  is distance non-increasing and we may simply take  $\delta = \varepsilon$ .

For Part (2), suppose that  $g(y) \in \widehat{X}$ . Then any  $x \in Y$  may be joined to  $y$  via a rectifiable path  $c$ . Since  $\phi \circ g$  is distance non-increasing,  $\phi \circ c$  is rectifiable, which shows that the distance between  $g(x)$  and  $g(y)$  is finite, i.e.  $g(x) \in \widehat{X}$ . Choosing the length of  $c$  to be close to  $d_Y(x, y)$  and applying the definition of the distance in  $\widehat{X}$  shows that  $g$  is distance non-increasing into  $\widehat{X}$ .

For Part (3), let  $c$  be a  $\rho$ -path in  $X$ . Then the unique lift of  $c$  to the basepoint (which lies in  $\widehat{X}$ ) satisfies the conditions of the second part of this theorem, and so the unique lift of  $c$  to  $\widetilde{X}$  to the basepoint must be a path in  $\widehat{X}$ . We have already observed that  $\widehat{\phi}$  is length preserving, so the lift has the same length. If there were

another lift of  $c$  to  $\widehat{X}$  then since the inclusion of  $\widehat{X}$  into  $\widetilde{X}$  is continuous there would be two lifts of  $c$  in  $\widetilde{X}$ , a contradiction, showing that  $\widehat{\phi}$  is a URL-map. Similarly, any 1-Lipschitz homotopy at the basepoint lifts uniquely to  $\widehat{X}$ , finishing the proof of the third part.

For the fourth part, let  $(z_j)_{j=1}^\infty$  be a Cauchy sequence in  $\widehat{X}$ . Letting  $x_j := \phi(z_j) \in X$ , the fact that  $\phi$  is distance non-increasing implies that  $\{x_j\}$  is Cauchy with limit  $x$ . Let  $\{\gamma_j\}$  be a collection of paths such that  $\gamma_j$  joins  $x_j$  to  $x$  and  $L(\gamma_j) < 2d(x_j, x)$ . Define  $\alpha_j$  to be the unique lift of  $\gamma_j$  starting at  $z_j$ . Since  $\phi$  is length preserving, if  $y_j$  denotes the endpoint of  $\alpha_j$ , for any  $i, j$  we have

$$d_{\widehat{X}}(y_i, y_j) \leq 2(d_X(x_j, x) + d_X(x_i, x))$$

and it follows from the convergence of  $\{x_j\}$  that  $\{y_j\}$  is Cauchy. Moreover, since the inclusion of  $\widehat{X}$  into  $\widetilde{X}$  is uniformly continuous,  $\{y_i\}$  is also Cauchy in  $\widetilde{X}$ . Now  $\{y_j\}$  is actually a sequence in  $\phi^{-1}(x)$ , which is an orbit of the action of  $\delta(X)$ . But this orbit is complete (with the metric induced by  $\widetilde{X}$ ) by Proposition 36 of [30] and therefore  $y_j \rightarrow y \in \widetilde{X}$ . Without loss of generality we may suppose that  $d(y_j, y_{j+1}) < 2^{-j}$  for all  $j$ . Let  $c_j$  be a path joining  $y_j$  and  $y_{j+1}$  of length less than  $2^{-j}$  in  $\widehat{X}$ . We may parameterize  $c_1$  on  $[0, \frac{1}{2}]$ ,  $c_2$  on  $[\frac{1}{2}, \frac{3}{4}]$ , and so on. The concatenation of these paths is a distance decreasing path from  $[0, 1]$  into  $\widehat{X}$  and hence a uniformly continuous path into  $\widetilde{X}$ . Since  $y_j \rightarrow y$  in  $\widetilde{X}$ , this path has a unique continuous extension to a path  $c : [0, 1] \rightarrow \widetilde{X}$  from  $y_1$  to  $y$ . Moreover, since  $\widehat{\phi}$  is length preserving,  $\widehat{\phi} \circ c$  is a path of length at most 1 in  $X$ . It now follows that  $y \in \widehat{X}$  and  $y_j \rightarrow y$  in  $\widehat{X}$ . By definition,  $d(z_j, y_j) \rightarrow 0$  and hence  $z_j \rightarrow y$  in  $\widehat{X}$  as well.

For the fifth part, suppose that  $g \in \delta(X)$  stabilizes  $\widehat{X}$ . Then  $*$  and  $g(*)$  are joined by an path  $\alpha$  such that  $\phi(\alpha)$  is rectifiable. Since  $\phi(g(*)) = \phi(*) = *$ ,  $\phi(\alpha)$  is a loop representing an element  $[\alpha] \in \mu(X)$  with  $\lambda([\alpha]) = g$ . On the other hand, suppose that  $g \in \mu(X)$  and let  $\alpha$  be a rectifiable loop representing  $g$ . Then  $*$  and  $g(*)$  are joined by  $\alpha_L$ . Now if  $x \in \widehat{X}$ , we may join  $*$  to  $x$  by a path  $\beta$  such that  $\phi \circ \beta$  is rectifiable. Now  $g \circ \beta$  joins  $g(*)$  and  $g(x)$ , and  $\phi \circ g \circ \beta = g \circ \beta$  is rectifiable. Therefore the concatenation  $\gamma$  of  $\alpha_L$  and  $g \circ \beta$  is a path joining  $*$  and  $g(x)$  such that  $\phi \circ \gamma$  is rectifiable. Therefore  $g(x) \in \widehat{X}$  and we have shown that  $\mu(X)$  is the stabilizer of  $\widehat{X}$ . The fact that  $\mu(X)$  acts isometrically follows from the fact that for any path  $\alpha$  in  $\widehat{X}$  and  $g \in \mu(X)$ ,  $\phi \circ \alpha = \phi \circ g \circ \alpha$ . Since  $\delta(X)$  acts freely, so does  $\mu(X)$ . Moreover, the orbits of  $\delta(X)$  are closed. Suppose  $p \in \widehat{X}$ . Since  $\mu(X)$  is the stabilizer of  $\widehat{X}$  in  $\delta(X)$ , the orbit  $\mu(X)p$  is the intersection of the orbit  $\delta(X)p$  with  $\widehat{X}$  and hence is closed in the subspace topology of  $\widehat{X} \subset \widetilde{X}$ . But we already know that the inclusion of  $\widehat{X}$  with the length is continuous, hence the orbits of  $\mu(X)$  are closed in this topology. Since  $\widehat{\phi}$  is already a metric quotient, the final part of the fifth statement is simply the observation that the orbits of  $\mu(X)$  are exactly the pre-images of points with respect to  $\widehat{\phi}$ .  $\square$

**Corollary 55.** *There is a unique (up to basepoint choice) URL-map  $\theta : \overline{X} \rightarrow \widehat{X}$  such that  $\overline{\phi} = \widehat{\phi} \circ \theta$ . Moreover,*

- (1) *The restriction  $\theta^* : \Lambda(X) \rightarrow \mu(X)$  is a surjective homomorphism.*
- (2) *If  $X$  is uniformly 1-dimensional then  $\theta$  is an isometry and  $\theta^*$  is an isomorphism.*

*Proof.* The first part is a corollary of Theorems 5 and 54. The fact that  $\theta^*$  is a homomorphism follows from the fact that the cancelled concatenation of two paths is homotopic to the concatenation. Now suppose that  $\gamma \in \mu(X)$ . Take a  $\rho$ -path  $c$

joining  $*$  and  $\gamma(*)$ . Then  $\widehat{\phi} \circ c \in \Lambda(X)$  with  $\theta^*(\widehat{\phi} \circ \gamma) = \gamma$  by definition, proving that  $\theta^*$  is surjective.

To prove the last part we need only show that  $\theta$  is an injection when  $X$  is uniformly 1-dimensional. But if  $\theta(c) = \theta(d)$  then the lifts  $c_L$  and  $d_L$  must have the same endpoint. The compositions  $c', d'$  of these paths with the inclusion of  $\widehat{X}$  into the simply connected uniformly one-dimensional space  $\widetilde{X}$  also have the same endpoints and hence must be fixed-endpoint homotopic by Corollary 14. But then  $c = \phi \circ c'$  and  $d = \phi \circ d'$  are also fixed-endpoint homotopic. By Corollary 12,  $c$  and  $d$  must be identical.  $\square$

**Remark 56.** *The kernel  $K$  of  $\theta^*$  is of obvious interest, since  $\pi_1^K(X) = \Lambda(X)/K$  is isomorphic to  $\mu(X)$ . Now  $K$  consists of all elements of  $\Lambda(X)$  that lift as loops via  $\widehat{\phi}$ , and so  $K = H_L$  if and only if  $\widehat{\phi}$  is  $H_L$ -simply connected.*

**Example 57.** *The infinite torus  $\mathbb{T}^\infty = S^1 \times S^1 \times \cdots$  can be metrized in the following natural way (cf. [7]). Give each  $S^1$  a geodesic metric such that the diameters of the factors are square summable, and apply the geodesic product metric to  $\mathbb{T}^\infty$  (this is not the “topologist’s” product metric, but the one that generalizes the usual Euclidean or Riemannian product metrics, and requires square summability of the diameters). For this particular metric we already considered the metric core in [7], although we did not have a general construction at the time and did not refer to it as the metric core. In fact the metric core is simply separable Hilbert space  $l^2$  consisting of all square summable sequences inside  $\widetilde{\mathbb{T}^\infty} = \mathbb{R} \times \mathbb{R} \times \cdots$ . Note that as is seen from the results in [7] there are non-rectifiable 1-parameter subgroups in  $\mathbb{T}^\infty$  that cannot be lifted to  $l^2$  (although they do, of course, lift to  $\widetilde{\mathbb{T}^\infty}$ ).*

## 6. APPENDIX

The uniform universal covering (UU-covering) was defined in [6], and provides an analog of the traditional universal covering for uniform spaces that are not necessarily semi-locally simply connected or even locally pathwise connected. In this appendix we will sketch the construction and results of [6] in the simplified setting of metric spaces, while also establishing simplified notation and providing a reference for those primarily interested in geodesic spaces as opposed to uniform spaces in full generality.

Let  $X$  be an arbitrary metric space. An  $\varepsilon$ -chain in  $X$  is a finite ordered sequence of points  $\{x_0, \dots, x_n\}$  such that for each  $i$ ,  $d(x_i, x_{i+1}) < \varepsilon$ ; the  $\varepsilon$ -chain is an  $\varepsilon$ -loop if  $x_0 = x_n$ . Two  $\varepsilon$ -chains having the same pair of endpoints are called  $\varepsilon$ -homotopic if one can be obtained from the other via a finite number of steps, each of which involves either adding or taking away a single point, maintaining an  $\varepsilon$ -chain at each step. The  $\varepsilon$ -homotopy equivalence class of an  $\varepsilon$ -chain  $\gamma$  is denoted by  $[\gamma]_\varepsilon$ . We may choose any basepoint  $*$  in  $X$ ; all theorems that follow are independent of the choice of basepoint, and when dealing with maps between spaces we may always assume that the map takes the basepoint of one space to the basepoint of the other.

The set of all  $\varepsilon$ -homotopy equivalence classes  $[\gamma]_\varepsilon$  of  $\varepsilon$ -chains starting at the basepoint  $*$  is denoted by  $X_\varepsilon$ . We denote by  $\delta_\varepsilon(X)$  the  $\varepsilon$ -deck group of  $X$ , which consists of  $\varepsilon$ -homotopy classes of  $\varepsilon$ -loops based at  $*$ , with the group operation induced by concatenation, which also acts as a group of bijections on  $X_\varepsilon$  via concatenation. Moreover,  $X$  is naturally identified with the orbit space  $\delta_\varepsilon(X) \backslash X_\varepsilon$ .

If  $0 < \delta < \varepsilon$  then every  $\delta$ -chain (resp. homotopy) may be considered as an  $\varepsilon$ -chain (resp. homotopy) and therefore there is a well-defined function  $\phi_{\varepsilon\delta} : X_\delta \rightarrow X_\varepsilon$  defined by  $\phi_{\varepsilon\delta}([\gamma]_\delta) = [\gamma]_\varepsilon$ . Since  $\phi_{\varepsilon\delta} = \phi_{\varepsilon\alpha} \circ \phi_{\alpha\delta}$  whenever  $0 < \delta < \alpha < \varepsilon$ , we have an inverse system  $(X_\varepsilon, \phi_{\varepsilon\delta})$  indexed by the positive reals with the reverse ordering. The inverse limit of this system is denoted by  $\widetilde{X}$ . Roughly speaking, elements of  $\widetilde{X}$

may be thought of as collections of discrete homotopy equivalence classes of finer and finer chains (rather than homotopy classes of paths used to construct the traditional universal covering). The natural projection  $\phi : \tilde{X} \rightarrow X$  takes an element  $([\gamma_\varepsilon]_\varepsilon)$  of  $\tilde{X}$  to the common endpoint  $x$  of all of the chains  $\gamma_\varepsilon$ . The set of  $([\gamma_\varepsilon]_\varepsilon) \in \tilde{X}$  such that each  $\gamma_\varepsilon$  is an  $\varepsilon$ -loop forms a group  $\delta(X)$  with respect to concatenation, called the *uniform fundamental group*. (In [6] we denoted this group by  $\delta_1(X)$  and called it the “deck group”.)  $\delta(X)$  acts freely on  $\tilde{X}$  by concatenation.

Now suppose that  $X$  is a length space. The length metric of  $X$  may be lifted to a length metric on  $X_\varepsilon$  in such a way that the bonding maps  $\phi_{\varepsilon\delta}$  are local isometries and traditional covering maps. The projection  $\phi : \tilde{X} \rightarrow X$  is surjective and is called the UU-covering of  $X$ , although it is not a traditional covering map.

**Notation 58.** *In any countable inverse limit construction, one may use a cofinal sequence of indices to obtain the inverse limit. For simplicity we will often use the sequence  $2^{-i}$  to index the system for  $i = 1, 2, \dots$ , and let  $X := X_0$ ,  $X_i := X_{2^{-i}}$ ,  $\phi_{ij} := \phi_{2^{-i}2^{-j}}$ ,  $\phi_i := \phi$ , and  $\delta_i(X) := \delta_{2^{-i}}(X)$ . We will denote elements of  $\tilde{X}$  as sequences  $(x_i)$  with  $x_i \in X_i$ .*

While  $\tilde{X}$  is metrizable, there is no hope of finding a compatible length on  $\tilde{X}$  because this space may not be pathwise connected. However, there is a metric  $\tilde{d}$  on  $\tilde{X}$  compatible with the inverse limit uniform structure such that  $\delta(X)$  acts isometrically and the orbits of  $\delta(X)$  in  $\tilde{X}$  are closed. Moreover, the orbit space  $\delta(X) \backslash \tilde{X}$  with the quotient metric is uniformly homeomorphic to  $X$  via the mapping  $x \mapsto \phi^{-1}(x)$ .

If  $X$  is a compact geodesic space that is semilocally simply connected then the fundamental inverse system stabilizes at some sufficiently large  $i$ , and the projection  $\phi_{\infty i} : X_i \rightarrow X$  is the traditional universal covering of  $X$ , and  $\delta(X) = \delta_i(X)$  is the fundamental group of  $X$ .

The UU-covering  $\phi : \tilde{X} \rightarrow X$  has the following lifting property. In [6] we defined a notion of “universal” uniform space that is an analog of simply connected in the traditional setting. If  $Y$  is universal and  $f : Y \rightarrow X$  is uniformly continuous, then there is a unique uniformly continuous function  $f_L : Y \rightarrow \tilde{X}$  such that  $\phi \circ f_L = f$  and  $f_L(*) = *$ . The function  $f_L$  will be called the *lift* of  $f$ . Since real segments and their cartesian products are universal, one may lift paths and homotopies to  $\tilde{X}$ . There is a natural homomorphism  $\lambda : \pi_1(X) \rightarrow \delta(X)$  defined as follows: If  $\alpha$  is a loop representing an element of  $\pi_1(X)$  based at  $*$  then  $\lambda([\alpha])$  is the unique element of  $\delta(X)$  that carries the basepoint  $*$  to the endpoint of  $\alpha_L$ . This homomorphism is injective if and only if the pathwise connected component  $P$  of  $*$  in  $\tilde{X}$  is simply connected, and surjective if and only if  $\tilde{X}$  is pathwise connected.

Although  $\tilde{X}$  need not be pathwise connected, from the results of [6] we know that the pathwise connected component of  $\tilde{X}$  is dense in  $\tilde{X}$  and in particular  $\tilde{X}$  is connected. Note that  $\phi_i : \tilde{X} \rightarrow X_i$  is the UU-covering of  $X_i$  for any  $i$ .

The UU-covering also has an induced mapping property: If  $f : X \rightarrow Y$  is uniformly continuous then there is a unique (basepoint preserving) uniformly continuous function  $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$  such that if  $\phi : \tilde{X} \rightarrow X$  and  $\psi : \tilde{Y} \rightarrow Y$  are the UU-coverings,  $f \circ \phi = \psi \circ \tilde{f}$ . The lift is functorial and the restriction of  $\tilde{f}$  to  $\delta(X) \subset \tilde{X}$  is a homomorphism  $f_* : \delta(X) \rightarrow \delta(Y) \subset \tilde{Y}$ .

Note that the mappings  $\phi_{ij} : X_j \rightarrow X_i$  are by construction URL-maps.

**Remark 59.** *The UU-covering map of any length space is also light; in fact point pre-images are inverse limits of discrete spaces and so are totally disconnected. Note that this statement, in particular, is also true in the more general setting of coverable uniform spaces described in [6].*

We proved in Proposition 92 and Theorem 93 from [6] the following:

**Proposition 60.** *If  $X$  is a coverable uniform uniformly one-dimensional space, then  $\tilde{X}$  is simply connected and contains no topological circles, and the homomorphism  $\lambda : \pi_1(X) \rightarrow \delta(X)$  is injective.*

**Remark 61.** *As we showed in [6], every connected, locally arcwise connected compact uniform space, hence any Peano continuum  $X$ , is coverable. But one may also apply the Bing-Moise theorem to obtain a compatible geodesic metric and use the construction described in this appendix.*

## REFERENCES

- [1] P.S. Alexandroff, B.A.Pasynkov, Introduction to dimension theory (Russian), Moscow, 1973.
- [2] R. D. Anderson, A continuous curve admitting monotone open maps onto all locally connected metric continua, abstract in BAMS 62 (1956) 264.
- [3] R. D. Anderson, Open mappings of continua, Summer Institute on Set Theoretic Topology, Amer. Math. Soc., Providence, R. I., 1958.
- [4] P. D. Andreev and V. N. Berestovskii, Dimensions of  $\mathbb{R}$ -trees and non-positively curved self-similar fractal spaces, (Russian). Mat. Trudy 9 (2006), No. 2, 3-22. Engl.transl.: Siberian Adv. Math., 2007, v. 17, No. 2, 1-12.
- [5] V.N. Berestovskii and L. Guijarro, A Metric Characterization of Riemannian Submersions, Annals of Global Analysis and Geometry 18 (2000) 577-588.
- [6] V. Berestovskii and C. Plaut, Uniform universal covers of uniform spaces, Top. Appl. 154 (2007) 1748-1777.
- [7] V. Berestovskii, C. Plaut, and C. Stallmann, Geometric groups I, Trans. Amer. Math. Soc. 361 (1999) 1403-1422.
- [8] M. Bestvina,  $\mathbb{R}$ -trees in topology, geometry and group theory *Handbook of Geometric Topology*, edited by R. Daverman and R. Sher, Elsevier, Amsterdam, 2002, 55-91.
- [9] R. H. Bing, Partitioning a set, Bull. Amer. Math. Soc. 55 (1949) 1101-1110.
- [10] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften 319, Springer-Verlag, Berlin, Heidelberg, New York, 1999.
- [11] Cannon, J. and Conner, G., On the fundamental groups of one-dimensional spaces, Topology Appl. 153 (2006), no. 14, 2648-2672.
- [12] M. Curtis and M. Fort, Jr., The fundamental group of one-dimensional spaces, Proc. Amer. Math. Soc. 10 (1959) 140-148.
- [13] M. Curtis and M. Fort, Jr., Singular homology of one-dimensional spaces, Ann. of Math. (2) 69 (1959) 309-313.
- [14] R. Deming, Some point-set properties and the edge path group of a generalized uniform space, TAMS 130 (1968) 387-405.
- [15] A. Dyubina and I. Polterovich, Explicit constructions of universal  $\mathbb{R}$ -trees and asymptotic geometry of hyperbolic spaces, Bull. Lon. Math. Soc. 33 (2001) 727-734.
- [16] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Birkhauser, Boston, Basel, Berlin, 1999.
- [17] W. Hurewicz, and H. Wallman, Dimension theory, Princeton, 1948.
- [18] J. Isbell, *Uniform Spaces*, Mathematical Surveys 12, American Mathematical Society, Providence, RI, 1964.
- [19] L. V. Keldyš, Example of a one-dimensional continuum with a zero-dimensional and interior mapping onto a square (Russian), Doklady Akad. Nauk SSSR (N.S.) 97 (1954) 201-204.
- [20] A. N. Kolmogorov, Über offene abbildungen, Ann. of Math. (2) 38 (1937) 36-38.
- [21] K. Kuratowski, *Topology*, Academic Press. New York and London, 1968.
- [22] P. Hajlasz and J. Tyson, in preparation 2008.
- [23] J. Mayer, J. Nikiel, and L. Oversteegen, Universal spaces for  $\mathbb{R}$ -trees, TAMS 334 (1992) 411-432.
- [24] K. Menger, Untersuchungen iiber allgemeine Metrik, Math. Ann. vol. 100 (1928) 75-163.
- [25] J. Milnor, A note on curvature and the fundamental group, J. Diff. Geo. 2 (1968) 1-7.
- [26] E. Moise, Grille decomposition and convexification theorems for compact locally connected continua, Bull. Amer. Math. Soc. vol. 55 (1949) 1111-1121.
- [27] J. W. Morgan, *Deformations of algebraic and geometric structures*, CBMS Lectures, UCLA, summer 1986.
- [28] J. W. Morgan and P. Shalen, Valuations, trees, and degenerations of hyperbolic structures, I, Ann. Math 120 (1984) 401-476.

- [29] C. Plaut, Metric spaces of curvature  $\geq k$ , Chapter 16, *Handbook of Geometric Topology*, edited by R. Daverman and R. Sher, Elsevier, Amsterdam, 2002.
- [30] C. Plaut, Quotients of uniform spaces, *Top. Appl.* 153 (2006) 2430-2444.
- [31] E. Spanier, *Algebraic Topology*, Springer, Berlin and New York, 1966.
- [32] J. Tits, A “theorem of Lie-Kolchin” for trees, in *Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin*, edited by H. Bass, P. Cassidy, and J. Kovacic, Academic Press, New York, 1977.
- [33] C. Sormani and G. Wei, Hausdorff Convergence and Universal Covers, *TAMS* 353 (2001) 3585-3602.
- [34] C. Sormani and G. Wei, Universal covers for Hausdorff limits of noncompact spaces, *TAMS* 356 (2003) 1233-1270.
- [35] C. Sormani and G. Wei, The covering spectrum of a compact length space, *Journal of Differential Geometry* 66 (2004) 647-689.
- [36] C. Sormani and G. Wei, The cut-off covering spectrum, preprint, arXiv:0705.3822, 2008.
- [37] D. Wilson, Open mappings of the universal curve onto continuous curves, *TAMS* 168 (1972) 497-515.
- [38] W. Wylie, Noncompact manifolds with nonnegative Ricci curvature, *J. Geom. Analysis* 16 (2006) 535-550.

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